Numerical analysis of lattice Boltzmann schemes

General results, monotonicity/convergence for a non-linear conservation law, and equilibrium boundary conditions

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26th of June, 2025

Séminaire SISMA au LIHPC

Introduction

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Lattice Boltzmann schemes

Fast and "physical" alternative [McNamara & Zanetti, '**88**], [Higuera & Jiménez, '**89**] to traditional numerical methods for PDEs (conservation laws):

- Incompressible NS
- Hyperbolic systems
- ...

Traditional numerical methods (FD, FV, FE, etc.)–way of doing

N PDEs $\mapsto$ Discretization and N numerical schemes (or unknowns)

Linear 1D transport ($N = 1$)

$$\partial_t u(t,x) + V\partial_x u(t,x) = 0 \quad \mapsto \quad \frac{\mathsf{u}_j^{n+1} - \mathsf{u}_j^{n-1}}{2\Delta t} + V\frac{\mathsf{u}_{j+1}^{n} - \mathsf{u}_{j-1}^{n}}{2\Delta x} = 0.$$

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Lattice Boltzmann–way of doing

Discretization and scheme on $q > N$ unknowns $\mapsto$ N PDEs

$\Delta t\mathbb{N} \times \Delta x\mathbb{Z}^d$

collide (local, non-linear)
and stream (non-local, shift, linear)

Outline of the talk

Introduction

Similar to transport-projection/kinetic schemes, e.g. [Brenier, '**83**] et [Bouchut, '**03**], but:

- Uniform Cartesian grids (Δt and Δx) and well-chosen (discrete) velocities.
- We avoid projecting on the equilibrium (kinetic unknowns really exist).
- Several "relaxation parameters" can be considered.

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Two-unknowns scheme ($q = 2$)

$$\partial_t u + \partial_x \varphi(u) = 0 \rightarrow \begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + \lambda^2 \partial_x u = \frac{1}{\epsilon}(\varphi(u) - v) \end{cases} \leftrightarrow \partial_t f^{\pm} \pm \lambda \partial_x f^{\pm} = \frac{1}{\epsilon}\left(\frac{u}{2} \pm \frac{\varphi(u)}{2\lambda} - f^{\pm}\right)$$

- Collision: $\partial_t u = 0$ and $\partial_t v = \frac{1}{\epsilon}(\varphi(u) - v)$. We have the exact solution!

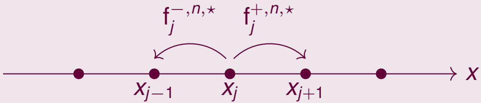
$$\mathrm{u}_j^{n,\star} = \mathrm{u}_j^n, \qquad \mathrm{v}_j^{n,\star} = (1 - e^{-\Delta t/\epsilon})\mathrm{v}_j^n + e^{-\Delta t/\epsilon}\varphi(\mathrm{u}_j^n) \xrightarrow{\epsilon \to 0^+} \varphi(\mathrm{u}_j^n).$$

  We do not do this: we replace $e^{-\Delta t/\epsilon}$ by $\omega \in (0, 2]$.

$$\mathrm{u}_j^{n,\star} = \mathrm{u}_j^n, \qquad \mathrm{v}_j^{n,\star} = (1 - \omega)\mathrm{v}_j^n + \omega\varphi(\mathrm{u}_j^n).$$

- Transport: $\partial_t f^{\pm} \pm \lambda \partial_x f^{\pm} = 0$. We pick $\lambda = \Delta x / \Delta t$ and use upwind schemes:

$$\mathrm{f}_j^{\pm,n+1} = \mathrm{f}_{j\mp 1}^{\pm,n,\star}.$$



Theoretical results by unknowns-elimination

Advancements in the understanding of these schemes

This is the way it is implemented (collision and transport diagonal in different bases).

For the analysis, we rewrite on u and v and employ the forward time-shift z:

$$\mathbf{m}^{n+1}(\boldsymbol{x}) = \begin{pmatrix} \mathrm{u}^{n+1}(\boldsymbol{x}) \\ \mathrm{v}^{n+1}(\boldsymbol{x}) \end{pmatrix} = z\mathbf{m}^{n}(\boldsymbol{x}) = \mathbf{A}\mathbf{m}^{n}(\boldsymbol{x}) + \mathbf{B}\boldsymbol{m}^{\mathrm{eq}}(\mathrm{m}_1^n(\boldsymbol{x})), \qquad \boldsymbol{x} \in \Delta x \mathbb{Z}^d.$$

Without boundaries or with periodic BC[1]:

1. Consistency
2. Stability
3. Initialisation

[1] [Numer. Math., '**22**], [M2AN, '**23**], [JCP, '**24**], and [CAMWA, '**24**]

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	Lattice Boltzmann	→	Finite Difference
Unknowns:	$\begin{pmatrix} \mathrm{m}_1^n \\ \mathrm{m}_2^n \\ \vdots \\ \mathrm{m}_q^n \end{pmatrix}$	$\mapsto$	$(\mathrm{m}_1^n, \mathrm{m}_1^{n-1}, \cdots, \mathrm{m}_1^{n-q+1})$
Scheme:	$(z\boldsymbol{Id}_q - \mathbf{A})\mathbf{m}^n = \mathbf{B}\boldsymbol{m}^{\mathrm{eq}}(\mathrm{m}_1^n)$	$\mapsto$	$\det(z\boldsymbol{Id}_q - \mathbf{A})\mathrm{m}_1^n = \boldsymbol{e}_1^{\mathsf{T}}\mathbf{adj}(z\boldsymbol{Id}_q - \mathbf{A})\mathbf{B}\boldsymbol{m}^{\mathrm{eq}}(\mathrm{m}_1^n)$

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On the two-unknowns scheme

Two-unknowns scheme ($q = 2$)

$$\det(z\boldsymbol{Id}_q - \mathbf{A})\mathrm{u}_j^{n-1} = \mathrm{u}_j^{n+1} + \tfrac{\omega-2}{2}(\mathrm{u}_{j-1}^n + \mathrm{u}_{j+1}^n) + (1-\omega)\mathrm{u}_j^{n-1},$$

$$\boldsymbol{e}_1^{\mathsf{T}}\mathbf{adj}(z\boldsymbol{Id}_q - \mathbf{A})\mathbf{B}\boldsymbol{m}^{\mathrm{eq}}(\mathrm{u}_j^{n-1}) = \tfrac{\omega\Delta t}{2\Delta x}(\varphi(\mathrm{u}_{j-1}^n) - \varphi(\mathrm{u}_{j+1}^n)).$$

- First interpretation: a θ–scheme.
  - Over-relax. $1 \leq \omega \leq 2$, a $\theta = (2-\omega)$–scheme between Lax-Friedrichs and leap-frog.
  - Under-relax. $0 \leq \omega \leq 1$, a $\theta = \omega$–scheme between Lax-Friedrichs and wave-leap-frog.
- Second interpretation: method-of-lines

$$\frac{\mathrm{u}_j^{n+1} + (\omega-2)\mathrm{u}_j^n + (1-\omega)\mathrm{u}_j^{n-1}}{\omega\Delta t} = -\frac{1}{\Delta x}(F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n)$$

$$\text{with} \qquad F_{j+\frac{1}{2}}^n = \frac{1}{2}(\varphi(\mathrm{u}_j^n) + \varphi(\mathrm{u}_{j+1}^n)) + \frac{\Delta x}{2\Delta t}\frac{2-\omega}{\omega}(\mathrm{u}_j^n - \mathrm{u}_{j+1}^n).$$

Curiosity: this flux when $\omega < 1$ combined with forward Euler for time-integration.

However, explicit formulas are of moderate interest when $q \gg 2$

Analyze LBM using FD (recycle), without computing the letter explicitly.

Convergence of scalar non-linear two-relaxation-times schemes on infinite domains

Problem and numerical scheme(s)

Preprint [Aregba-Driollet, B., *arXiv:2501.07934* ('**25**)]. Find (approximate) the unique weak entropy solution of:

$$\begin{cases} \partial_t u(t, \boldsymbol{x}) + \sum_{k=1}^{d} \partial_{x_k} \varphi_k(u(t, \boldsymbol{x})) = 0, & t > 0, \quad \boldsymbol{x} \in \mathbb{R}^d, \\ u(0, \boldsymbol{x}) = u^\circ(\boldsymbol{x}), & \qquad\quad \boldsymbol{x} \in \mathbb{R}^d, \end{cases}$$

with $\varphi_k \in C^1(\mathbb{R})$ and $u^\circ \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap \mathrm{BV}(\mathbb{R}^d)$.

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We consider $q = 1 + 2W$, where $W \in \mathbb{N}^*$, with

$$\boldsymbol{c}_1 = \boldsymbol{0}, \qquad \boldsymbol{c}_{2\ell} = -\boldsymbol{c}_{2\ell+1} \in \lambda \mathbb{Z}^d, \qquad \ell \in [\![1, W]\!].$$

Denote $\mathrm{u}_{\boldsymbol{j}}^n = \sum_{i=1}^{i=q} \mathrm{f}_{i,\boldsymbol{j}}^n$.
Relaxation (briefly $\mathrm{f}_{i,\boldsymbol{j}}^{n,\star} = \mathcal{R}_i(\mathrm{f}_{1,\boldsymbol{j}}^n, \dots, \mathrm{f}_{q,\boldsymbol{j}}^n)$):

$$\begin{aligned} \mathrm{f}_{1,\boldsymbol{j}}^{n,\star} &= (1 - \omega_{\mathrm{s}})\mathrm{f}_{1,\boldsymbol{j}}^n + \omega_{\mathrm{s}} f_1^{\mathrm{eq}}(\mathrm{u}_{\boldsymbol{j}}^n), \\ \mathrm{f}_{2\ell,\boldsymbol{j}}^{n,\star} &= \big(1 - \tfrac{1}{2}(\omega_{\mathrm{s}} + \omega_{\mathrm{a}})\big)\mathrm{f}_{2\ell,\boldsymbol{j}}^n + \tfrac{1}{2}(\omega_{\mathrm{s}} + \omega_{\mathrm{a}}) f_{2\ell}^{\mathrm{eq}}(\mathrm{u}_{\boldsymbol{j}}^n) - \tfrac{1}{2}(\omega_{\mathrm{s}} - \omega_{\mathrm{a}})(\mathrm{f}_{2\ell+1,\boldsymbol{j}}^n - f_{2\ell+1}^{\mathrm{eq}}(\mathrm{u}_{\boldsymbol{j}}^n)), \\ \mathrm{f}_{2\ell+1,\boldsymbol{j}}^{n,\star} &= \big(1 - \tfrac{1}{2}(\omega_{\mathrm{s}} + \omega_{\mathrm{a}})\big)\mathrm{f}_{2\ell+1,\boldsymbol{j}}^n + \tfrac{1}{2}(\omega_{\mathrm{s}} + \omega_{\mathrm{a}}) f_{2\ell+1}^{\mathrm{eq}}(\mathrm{u}_{\boldsymbol{j}}^n) - \tfrac{1}{2}(\omega_{\mathrm{s}} - \omega_{\mathrm{a}})(\mathrm{f}_{2\ell,\boldsymbol{j}}^n - f_{2\ell}^{\mathrm{eq}}(\mathrm{u}_{\boldsymbol{j}}^n)), \end{aligned}$$

Transport: $\mathrm{f}_{i,\boldsymbol{j}}^{n+1} = \mathrm{f}_{i,\boldsymbol{j}-\boldsymbol{c}_i/\lambda}^{n,\star}$.

Consistency and interest of having $\omega_{\mathrm{s}} \neq \omega_{\mathrm{a}}$

The scheme is consistent under the constraints

$$\sum_{i=1}^{q} f_i^{\mathrm{eq}}(\mathrm{u}) = \mathrm{u}, \qquad \sum_{i=1}^{q} c_{i,k} f_i^{\mathrm{eq}}(\mathrm{u}) = \sum_{\ell=1}^{W} c_{2\ell,k}(f_{2\ell}^{\mathrm{eq}}(\mathrm{u}) - f_{2\ell+1}^{\mathrm{eq}}(\mathrm{u})) = \varphi_k(\mathrm{u}),$$

under which, the modified equation up to second-order reads

$$\underbrace{\partial_t u(t,\boldsymbol{x}) + \sum_{k=1}^{d} \partial_{x_k} \varphi_k(u(t,\boldsymbol{x}))}_{\text{(target PDE)}} = \underbrace{\frac{2\Delta x}{\lambda}\Big(\frac{1}{\omega_{\mathrm{a}}} - \frac{1}{2}\Big) \times \text{(2nd-order diff. op.)}}_{\text{(numerical diffusion)}} + \mathcal{O}(\Delta x^2).$$

Informally, on $0 < \omega_{\mathrm{s}}, \omega_{\mathrm{a}} \leq 2$:

$$\begin{aligned} &\min \text{(numerical diffusion)}(\omega_{\mathrm{a}}) \qquad \text{with} \qquad \text{(numerical diffusion)}(\searrow), \\ &\max \text{“(stability)”}(\omega_{\mathrm{s}}, \omega_{\mathrm{a}}) \qquad \text{with} \qquad \text{“(stability)”}(\searrow, \searrow). \end{aligned}$$

Antinomic if $\omega_{\mathrm{s}} = \omega_{\mathrm{a}}$.

Main result

We further assume that (see [Natalini, '98])

$$f_i^{\mathrm{eq}}(\mathrm{u}) = \mathscr{L}_i \mathrm{u} + \sum_{k=1}^{d} \mathscr{N}_{i,k}\varphi_k(\mathrm{u}), \qquad i \in [\![1, q]\!],$$

with the symmetry conditions $\mathscr{L}_{2\ell} = \mathscr{L}_{2\ell+1}$ and $\mathscr{N}_{2\ell,k} = -\mathscr{N}_{2\ell+1,k}$.

Theorem (Convergence to the weak entropy solution)

Define $u_\infty := \|u^\circ\|_{L^\infty}$. *Assume that*

$$\text{(MC)} \qquad \begin{cases} \omega_{\mathrm{s}}\mathscr{L}_1 \geq \max(0, \omega_{\mathrm{s}} - 1), \\ \omega_{\mathrm{a}} \underbrace{\max_{\mathrm{u}\in[-u_\infty, u_\infty]} \left|\sum_{k=1}^{d} \mathscr{N}_{2\ell,k}\varphi_k'(\mathrm{u})\right|}_{\approx \textit{Courant number}} \leq \omega_{\mathrm{s}}\mathscr{L}_{2\ell} + \frac{1}{2}\min(2 - \omega_{\mathrm{s}} - \omega_{\mathrm{a}}, 0, \omega_{\mathrm{a}} - \omega_{\mathrm{s}}). \end{cases}$$

Up to extract, there exists $\overline{\boldsymbol{f}}$ *such that* $\overline{\boldsymbol{f}}(t,\cdot) \in L^1(\mathbb{R}^d)$ *(plus other properties) such that*

$$\lim_{p\to+\infty} \|\boldsymbol{f}_{\Delta_p} - \overline{\boldsymbol{f}}\|_{L_t^\infty([0,T])L_{\boldsymbol{x}}^1} = 0,$$

and, setting $\overline{u} := \sum_{i=1}^{i=q} \overline{f}_i$, *then* $\lim_{p\to+\infty}\|u_{\Delta_p} - \overline{u}\|_{L_t^\infty([0,T])L_{\boldsymbol{x}}^1} = 0$. *Moreover,* $\overline{\boldsymbol{f}}(t,\boldsymbol{x}) = \boldsymbol{f}^{\mathrm{eq}}(\overline{u}(t,\boldsymbol{x}))$ *a.e. in* $\boldsymbol{x}$. *Finally,* $\overline{u}$ *is a the weak entropy solution of the PDE.*

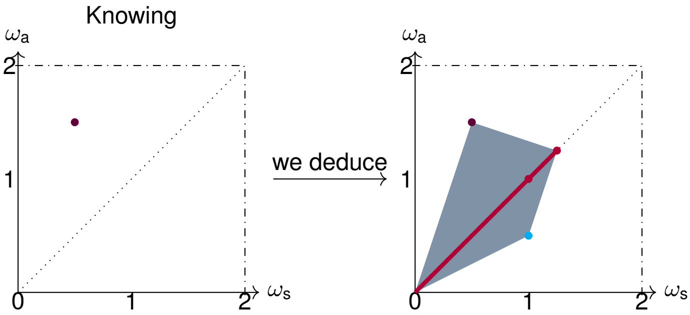
Steps of the proof

We remind that a time-step of the schemes reads $\mathsf{f}^{n+1}_{i,\boldsymbol{j}} = \mathcal{R}_i(\mathsf{f}^n_{1,\boldsymbol{j}-\boldsymbol{c}_i/\lambda}, \dots, \mathsf{f}^n_{q,\boldsymbol{j}-\boldsymbol{c}_i/\lambda})$.

1. Request that $\mathcal{R}_i(\nearrow, \dots, \nearrow)$ for all $i \in [\![1, q]\!]$: we find (MC).
2. The scheme preserves invariant (compact) sets (proof: mean-value theorem)

$$\mathsf{f}^n_{i,\boldsymbol{j}} \in [f^{\mathrm{eq}}_i(-u_\infty), f^{\mathrm{eq}}_i(u_\infty)], \quad \text{and} \quad \mathsf{u}^n_{\boldsymbol{j}} = \sum_{i=1}^{q} \mathsf{f}^n_{i,\boldsymbol{j}} \in [-u_\infty, u_\infty].$$

3. Equilibria are monotone functions (proof by contradiction): $\mathrm{d}_{\mathrm{u}} f^{\mathrm{eq}}_i(\mathrm{u}) \geq 0$ for $\mathrm{u} \in [-u_\infty, u_\infty]$, and other rigidity results on $(\omega_{\mathrm{s}}, \omega_{\mathrm{a}})$ fulfilling (MC).



4. ℓ^1 contractivity (proof: mean-value theorem and triangle inequality):

$$\sum_{i=1}^{q} |\mathcal{R}_i(\mathsf{g}_1, \dots, \mathsf{g}_q) - \mathcal{R}_i(\mathsf{f}_1, \dots, \mathsf{f}_q)| \leq \sum_{i=1}^{q} |\mathsf{g}_i - \mathsf{f}_i|,$$

hence L^1 contractivity with $\|u^\circ\|_{L^\infty} \leq u_\infty$, $\|v^\circ\|_{L^\infty} \leq u_\infty$:

$$\|\boldsymbol{g}_\Delta^{n+1} - \boldsymbol{f}_\Delta^{n+1}\|_{L^1} \leq \|\boldsymbol{g}_\Delta^{n} - \boldsymbol{f}_\Delta^{n}\|_{L^1} \leq \|v^\circ - u^\circ\|_{L^1},$$

and equicontinuity

$$\|\boldsymbol{f}_\Delta(\tilde{t}, \cdot) - \boldsymbol{f}_\Delta(t, \cdot)\|_{L^1} \leq \lambda C(\tilde{t} - t + \Delta t)\mathrm{TV}(u^\circ).$$

and total variation estimates:

$$\mathrm{TV}(\boldsymbol{f}_\Delta^{n+1}) \leq \mathrm{TV}(\boldsymbol{f}_\Delta^{n}) \leq \cdots \leq \mathrm{TV}(\boldsymbol{f}_\Delta^{0}) \leq \mathrm{TV}(u^\circ),$$

5. Convergence to equilibrium:

$$\begin{aligned}\|\boldsymbol{f}_\Delta^{n} - \boldsymbol{f}^{\mathrm{eq}}(u_\Delta^n)\|_{L^1} &\leq C\Delta x \mathrm{TV}(u^\circ) \frac{\max(|1-\omega_{\mathrm{s}}|, |1-\omega_{\mathrm{a}}|)^n - 1}{\max(|1-\omega_{\mathrm{s}}|, |1-\omega_{\mathrm{a}}|) - 1} \\ &\leq \frac{C}{1 - \max(|1-\omega_{\mathrm{s}}|, |1-\omega_{\mathrm{a}}|)} \Delta x \mathrm{TV}(u^\circ).\end{aligned}$$

Compactness and extraction

All the previous result ensure, upon extracting, there exists $\overline{\boldsymbol{f}}$ such that $\overline{\boldsymbol{f}}(t, \cdot) \in L^1(\mathbb{R}^d)$ such that

$$\lim_{p \to +\infty} \|\boldsymbol{f}_{\Delta_p} - \overline{\boldsymbol{f}}\|_{L^\infty_t([0,T])L^1_{\boldsymbol{x}}} = 0.$$

Upon extracting again, we know that $L^1_{\boldsymbol{x}}$-convergence implies point-wise convergence a.e., thus $\overline{f}_i(t, \boldsymbol{x}) \in [f_i^{\text{eq}}(-u_\infty), f_i^{\text{eq}}(u_\infty)]$, thus $\overline{u}(t, \boldsymbol{x}) \in [-u_\infty, u_\infty]$, a.e. in $\boldsymbol{x}$.

By the convergence to equilibrium, we deduce that $\overline{\boldsymbol{f}}(t, \boldsymbol{x}) = \boldsymbol{f}^{\text{eq}}(\overline{u}(t, \boldsymbol{x}))$ a.e. in $\boldsymbol{x}$.

Consistency with the weak form of the PDE

$$\Delta t \Delta x^d \sum_{n\in\mathbb{N}} \sum_{\boldsymbol{j}\in\mathbb{Z}^d} \frac{\mathsf{u}_{\boldsymbol{j}}^{n+1} - \mathsf{u}_{\boldsymbol{j}}^{n}}{\Delta t} \psi_{\boldsymbol{j}}^n = \Delta t \Delta x^d \sum_{n\in\mathbb{N}} \sum_{\boldsymbol{j}\in\mathbb{Z}^d} \frac{\sum_{i=1}^{q} \mathcal{R}_i(\mathsf{f}_{1,\boldsymbol{j}-\boldsymbol{c}_i/\lambda}^n, \dots) - \mathsf{u}_{\boldsymbol{j}}^n}{\Delta t} \psi_{\boldsymbol{j}}^n.$$

By standard summations-by-parts:

$$\overbrace{\int_{\Delta t}^{+\infty}\int_{\mathbb{R}^d} u_\Delta(t,\boldsymbol{x}) \frac{\psi_\Delta(t-\Delta t, \boldsymbol{x}) - \psi_\Delta(t,\boldsymbol{x})}{\Delta t} \mathrm{d}\boldsymbol{x}\mathrm{d}t}^{\to -\iint \overline{u}\partial_t \psi} - \overbrace{\int_{\mathbb{R}^d} u_\Delta(0,\boldsymbol{x})\psi_\Delta(0,\boldsymbol{x})\mathrm{d}\boldsymbol{x}}^{\to \int u^{\circ}\psi(t=0)}$$

$$= \underbrace{\int_{0}^{+\infty}\int_{\mathbb{R}^d} \frac{1}{\Delta t}\Big(\sum_{i=1}^{q} \mathcal{R}_i(\boldsymbol{f}_\Delta(t,\boldsymbol{x}))\psi_\Delta(t, \boldsymbol{x} + \Delta x \boldsymbol{c}_i/\lambda) - u_\Delta(t,\boldsymbol{x})\psi_\Delta(t,\boldsymbol{x})\Big)\mathrm{d}\boldsymbol{x}\mathrm{d}t}_{F}.$$

For the flux term:

$$F \to \int_{0}^{+\infty}\int_{\mathbb{R}^d} \sum_{k=1}^{d}\sum_{\ell=1}^{W} c_{2\ell,k}(f_{2\ell}^{\mathrm{eq}}(\overline{u}(t,\boldsymbol{x})) - f_{2\ell+1}^{\mathrm{eq}}(\overline{u}(t,\boldsymbol{x})))\partial_{x_k}\psi(t,\boldsymbol{x})\mathrm{d}\boldsymbol{x}\mathrm{d}t$$

$$= \int_{0}^{+\infty}\int_{\mathbb{R}^d} \sum_{k=1}^{d} \varphi_k(\overline{u}(t,\boldsymbol{x}))\partial_{x_k}\psi(t,\boldsymbol{x})\mathrm{d}\boldsymbol{x}\mathrm{d}t,$$

Entropy inequalities

We utilize Krushkov kinetic entropies, see [Natalini, '98]: $s_i(\mathsf{f}_i) := |\mathsf{f}_i - f_i^{\mathrm{eq}}(\kappa)|$, for $\kappa \in \mathbb{R}$.

The discrete entropy inequality (on post-relaxation quantities [Caetano, Graille, Dubois, '24]), using the ℓ^1 contractivity of the relaxation is

$$\frac{\sum_{i=1}^{q} s_i(\mathsf{f}_{i,\boldsymbol{j}}^{n+1,\star}) - \sum_{i=1}^{q} s_i(\mathsf{f}_{i,\boldsymbol{j}}^{n,\star})}{\Delta t} \leq \frac{\sum_{i=1}^{q} s_i(\mathsf{f}_{i,\boldsymbol{j}-\boldsymbol{c}_i/\lambda}^{n,\star}) - \sum_{i=1}^{q} s_i(\mathsf{f}_{i,\boldsymbol{j}}^{n,\star})}{\Delta t}.$$

Things go well since we have proved that under (MC), equilibria are monotone and thus

$$\sum_{i=1}^{q} s_i(f_{\Delta,i}) \to \sum_{i=1}^{q} |f_i^{\mathrm{eq}}(\overline{u}) - f_i^{\mathrm{eq}}(\kappa)| = |\overline{u} - \kappa| \qquad \text{(Krushkov entropy)}$$

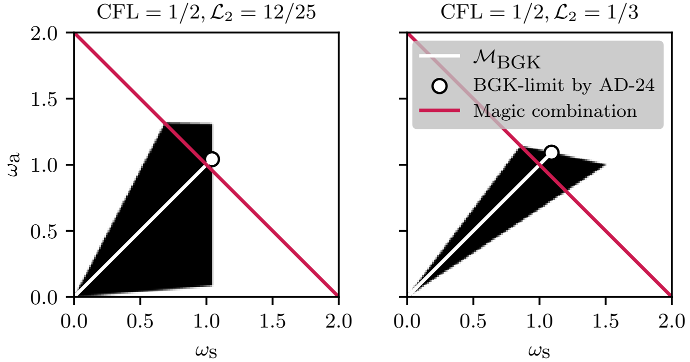
since we converge almost everywhere to the equilibrium. This rules the left-hand side of the weak entropy inequality in the limit.

The right-hand side is treated analogously.

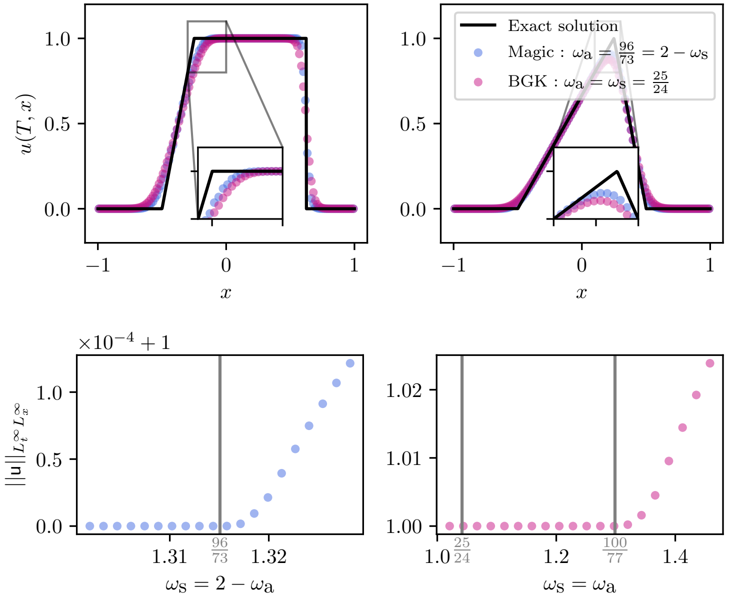
Take $d = 1$, $W = 1$, and $c_2 = \lambda$. By the consistency constraints

$$\mathscr{L}_1 = 1 - 2\mathscr{L}_2 \qquad \text{and} \qquad \mathscr{N}_2 = \frac{1}{2\lambda},$$

leaving $\mathscr{L}_2$ as a free parameter.
We take $\text{CFL} := \frac{1}{\lambda} \max_{u \in [-u_\infty, u_\infty]} |\varphi'(u)| = \frac{1}{2}$. We use a Burgers flux $\varphi(u) = u^2/2$ and use initial data within the interval $[0, 1]$. The numerical simulations are conducted on the domain $[-1, 1]$, equipped with periodic boundary conditions, until final time $T = \frac{1}{4}$.



Numerical experiments



Numerical experiments

Table: Errors and orders of convergence under magic combination ($\omega_s + \omega_a = 2$).

Δx	$\omega_a = 1$ Error	Ord.	$\omega_a = \frac{169}{146}$ Error	Ord.	$\omega_a = \frac{96}{73}$ Error	Ord.	$\omega_a = \frac{3}{2}$ Error	Ord.	$\omega_a = \frac{199}{100}$ Error	Ord.	$\omega_a = 2$ Error	Ord.
Indicator function initial datum: $u^{\circ}(x) = \mathbb{1}_{[0,1/2]}(\lvert x \rvert)$												
3.13E-02	1.49E-01		1.28E-01		1.10E-01		9.23E-02		1.36E-01		1.41E-01	
1.56E-02	9.26E-02	0.69	7.68E-02	0.73	6.37E-02	0.79	5.12E-02	0.85	9.87E-02	0.46	1.12E-01	0.36
7.81E-03	5.55E-02	0.74	4.50E-02	0.77	3.65E-02	0.81	2.85E-02	0.85	9.31E-02	0.08	1.23E-01	-0.13
3.91E-03	3.22E-02	0.78	2.59E-02	0.80	2.07E-02	0.82	1.59E-02	0.85	6.99E-02	0.42	1.14E-01	0.10
1.95E-03	1.84E-02	0.81	1.47E-02	0.82	1.16E-02	0.83	8.77E-03	0.85	4.24E-02	0.72	1.08E-01	0.08
9.77E-04	1.04E-02	0.83	8.21E-03	0.84	6.46E-03	0.85	4.81E-03	0.87	2.24E-02	0.92	1.06E-01	0.03
4.88E-04	5.80E-03	0.84	4.55E-03	0.85	3.56E-03	0.86	2.62E-03	0.88	1.12E-02	1.00	1.02E-01	0.06
2.44E-04	3.21E-03	0.86	2.50E-03	0.86	1.94E-03	0.87	1.41E-03	0.89	5.69E-03	0.98	1.01E-01	0.02
1.22E-04	1.76E-03	0.87	1.37E-03	0.88	1.05E-03	0.88	7.60E-04	0.90	2.86E-03	0.99	9.92E-02	0.02
6.10E-05	9.57E-04	0.88	7.39E-04	0.89	5.67E-04	0.89	4.06E-04	0.90	1.43E-03	1.00	9.87E-02	0.01
Hat function initial datum: $u^{\circ}(x) = (1 - 2\lvert x \rvert)\mathbb{1}_{[0,1/2]}(\lvert x \rvert)$												
3.13E-02	5.98E-02		4.53E-02		3.38E-02		2.28E-02		8.48E-03		8.66E-03	
1.56E-02	3.12E-02	0.94	2.32E-02	0.97	1.70E-02	1.00	1.12E-02	1.03	3.19E-03	1.41	3.28E-03	1.40
7.81E-03	1.59E-02	0.97	1.18E-02	0.98	8.51E-03	1.00	5.53E-03	1.01	1.21E-03	1.39	1.29E-03	1.35
3.91E-03	8.08E-03	0.98	5.92E-03	0.99	4.27E-03	1.00	2.75E-03	1.01	4.66E-04	1.38	5.12E-04	1.33
1.95E-03	4.07E-03	0.99	2.98E-03	0.99	2.14E-03	1.00	1.38E-03	1.00	1.80E-04	1.38	2.05E-04	1.32
9.77E-04	2.05E-03	0.99	1.49E-03	1.00	1.07E-03	1.00	6.87E-04	1.00	7.15E-05	1.33	8.31E-05	1.30
4.88E-04	1.03E-03	1.00	7.48E-04	1.00	5.36E-04	1.00	3.43E-04	1.00	2.84E-05	1.33	3.37E-05	1.30
2.44E-04	5.14E-04	1.00	3.74E-04	1.00	2.68E-04	1.00	1.72E-04	1.00	1.13E-05	1.33	1.37E-05	1.30
1.22E-04	2.57E-04	1.00	1.87E-04	1.00	1.34E-04	1.00	8.58E-05	1.00	4.53E-06	1.32	5.54E-06	1.30
6.10E-05	1.29E-04	1.00	9.37E-05	1.00	6.70E-05	1.00	4.29E-05	1.00	1.83E-06	1.31	2.25E-06	1.30

Convergence of scalar non-linear two-relaxation-times schemes with equilibrium boundary conditions

Problem

[Aregba-Driollet, B., *arXiv:2505.17535* ('**25**)]. Multi-dimensional, but I illustrate in 1D. The approach also works for system (not the proof)

$$\begin{aligned}
&\partial_t u(t,x) + \partial_x(\varphi(u(t,x))) = 0, && (t,x) \in (0,T)\times(0,1),\\
&u(0,x) = u^\circ(x), && x \in (0,1),\\
&u(t,x=0) = \tilde{u}_{◐}(t), && t \in (0,T),\\
&u(t,x=1) = \tilde{u}_{◑}(t), && t \in (0,T).
\end{aligned}$$

Definition of weak entropy solution by [Bardos, Leroux & Nédélec, '**79**]: for any $\kappa \in \mathbb{R}$ and any smooth $\psi \geq 0$

$$\begin{aligned}
&\int_0^T\int_0^1 |u(t,x)-\kappa| + \text{sgn}(u(t,x)-\kappa)((\varphi(u(t,x))-\varphi(\kappa))\partial_x\psi(t,x)\text{d}t\\
&+\int_0^T \text{sgn}(\tilde{u}_{◐}(t)-\kappa)(\varphi(\gamma_{◐}(u)(t))-\varphi(\kappa))\psi(t,0)\text{d}t\\
&-\int_0^T \text{sgn}(\tilde{u}_{◑}(t)-\kappa)(\varphi(\gamma_{◑}(u)(t))-\varphi(\kappa))\psi(t,1)\text{d}t + \int_0^1 |u^\circ(x)-\kappa|\psi(0,x)\text{d}x \geq 0.
\end{aligned}$$

Numerical scheme(s) and main result

Same as the previous point. For the boundary condition, for exemple

$$\mathsf{f}_{i,-1}^{n,\star} = f_i^{\text{eq}}(\tilde{\mathsf{u}}_{\mathbin{\scriptstyle\circ}}^n), \qquad \text{with} \qquad \tilde{\mathsf{u}}_{\mathbin{\scriptstyle\circ}}^n = \fint_{t^n}^{t^{n+1}} \tilde{u}_{\mathbin{\scriptstyle\circ}}(t)\mathrm{d}t$$

Theorem (Convergence to the weak entropy solution)

Define $u_\infty := \max(\|u^\circ\|_{L^\infty}, \|\tilde{u}_{\mathbin{\scriptstyle\circ}}\|_{L^\infty}, \|\tilde{u}_{\mathbin{\scriptstyle\bullet}}\|_{L^\infty})$. *Assume that*

$$\text{(MC)} \quad \begin{cases} \omega_{\mathrm{s}}\mathscr{L}_1 \geq \max(0, \omega_{\mathrm{s}} - 1), \\ \omega_{\mathrm{a}} \underbrace{\max_{\mathrm{u}\in[-u_\infty, u_\infty]} |\mathscr{N}_{2\ell}\varphi'(\mathrm{u})|}_{\approx \text{Courant number}} \leq \omega_{\mathrm{s}}\mathscr{L}_{2\ell} + \frac{1}{2}\min(2 - \omega_{\mathrm{s}} - \omega_{\mathrm{a}}, 0, \omega_{\mathrm{a}} - \omega_{\mathrm{s}}). \end{cases}$$

Up to extract, there exists $\overline{\boldsymbol{f}}$ *such that* $\overline{\boldsymbol{f}}(t,\cdot) \in L^1(\mathbb{R}^d)$ *(plus other properties) such that*

$$\lim_{p\to+\infty} \|\boldsymbol{f}_{\Delta_p} - \overline{\boldsymbol{f}}\|_{L_t^\infty(0,T)L_x^1(0,1)} = 0,$$

and, setting $\overline{u} := \sum_{i=1}^{i=q} \overline{f}_i$*, then* $\lim_{p\to+\infty}\|u_{\Delta_p} - \overline{u}\|_{L_t^\infty(0,T)L_x^1(0,1)} = 0$*. Moreover,* $\overline{\boldsymbol{f}}(t,x) = \boldsymbol{f}^{\text{eq}}(\overline{u}(t,x))$ *a.e. in* x*. Finally,* $\overline{u}$ *is a the weak entropy solution of the PDE.*

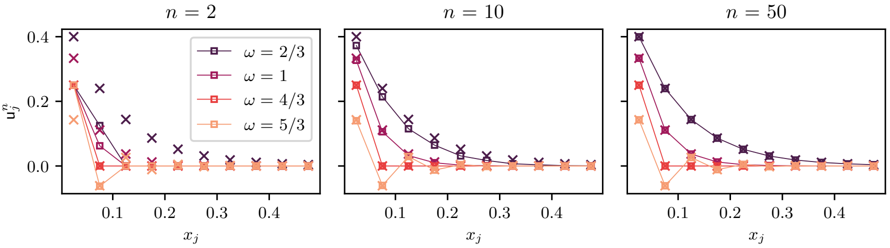
Numerical experiments

Two-velocities scheme solving, for $V < 0$:

$$
\begin{aligned}
&\partial_t u(t,x) + V\partial_x u(t,x) = 0, && (t,x) \in (0,\tfrac{1}{2}) \times (0,1),\\
&u(0,x) = 0, && x \in (0,1),\\
&u(t,1) = \tilde{u}_{\RIGHTcircle}(t) = 0, && t \in (0,\tfrac{1}{2}).
\end{aligned}
$$

On the left, we enforce the equilibrium corresponding to a constant $\tilde{\mathrm{u}}^{n}_{\LEFTcircle} = \tilde{\mathrm{u}}_{\LEFTcircle} = 1$.
The boundary layer (only for $\omega < 2$) can be described for n large enough:

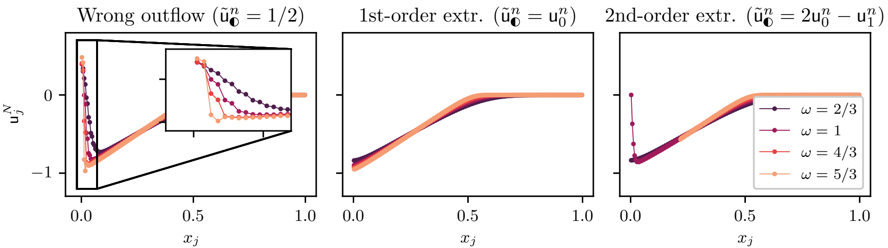
$$
\mathrm{u}_j^n \approx \begin{cases} \tilde{\mathrm{u}}_{\LEFTcircle} \frac{(2-\omega)(1+V/\lambda)}{2-\omega(1+V/\lambda)} \left(\frac{2-\omega+\omega V/\lambda}{2-\omega-\omega V/\lambda}\right)^j, & \text{if} \quad \omega \neq \frac{2}{1-V/\lambda},\\ \tilde{\mathrm{u}}_{\LEFTcircle} \frac{1+V/\lambda}{2} \delta_{j0}, & \text{if} \quad \omega = \frac{2}{1-V/\lambda}. \end{cases} \tag{1}
$$



Two-velocities scheme solving

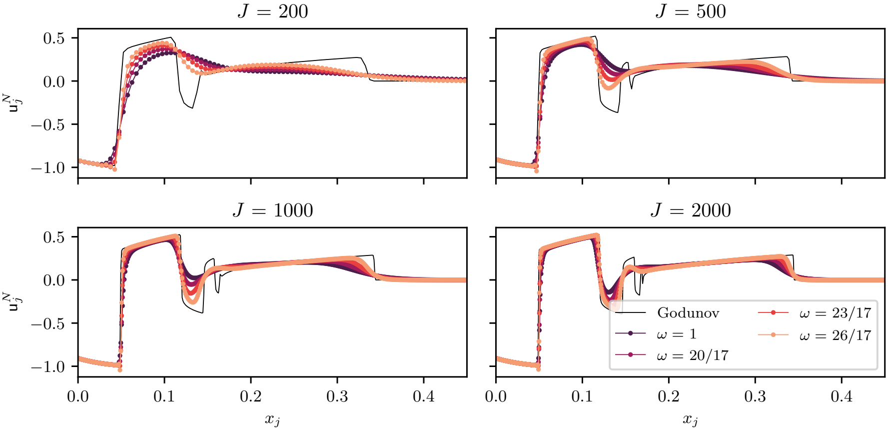
$$\partial_t u(t,x) + \partial_x(\tfrac{1}{2}u(t,x)^2) = 0, \qquad (t,x) \in (0,\tfrac{1}{2}) \times (0,1),$$
$$u(0,x) = u^{\circ}(x) = -\mathbb{1}_{(1/5,1/2)}(x), \qquad x \in (0,1),$$
$$u(t,1) = \tilde{u}_{◑}(t) = 0, \qquad t \in (0,\tfrac{1}{2}).$$

We propose a simulation with 200 cells, using $\lambda = 2$.



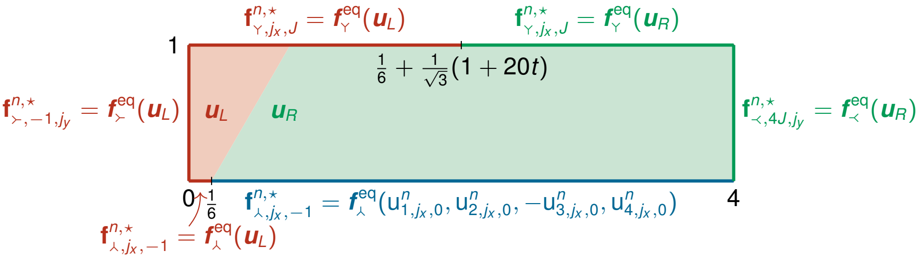
Numerical experiments

$$
\begin{aligned}
&\partial_t u(t,x) + \partial_x\left(\tfrac{1}{3}u(t,x)^3\right) = 0, && (t,x)\in(0,4)\times(0,1),\\
&u(0,x) = u^{\circ}(x) = 0, && x\in(0,1),\\
&u(t,0) = \tilde{u}_{◐}(t) = \sin(6t), && t\in(0,4),\\
&u(t,1) = \tilde{u}_{◑}(t) = 0, && t\in(0,4).
\end{aligned}
$$



Numerical experiments

Full Euler system. Double Mach 10 reflection [Woodward & Colella, '**84**]. Blended scheme by [Wissocq *et al.*, '**24**]



2nd order blended LBM - 480x120 gridpoints

Conclusions and perspectives

Conclusions and perspectives

What has been done

Convergence for non-linear scalar two-relaxation-times schemes without boundary and with equilibrium boundary conditions.

These boundary conditions work well outside the range of applicability of the theorem (non-monotone schemes and systems).

What still has to be done

- Understand why things work even when negative coefficients are present.
- *Systems* of conservation laws.
- Stability of more general boundary conditions (even just linear): ongoing.

Thank you for you attention!