Local subcell monolithic DG/FV scheme

François Vilar

Institut Montpelliérain Alexander Grothendieck, Université de Montpellier, France

Séminaire SISMA

Scalar Conservation Law (SCL)

- $\partial_t\, u(\mathbf{x}, t) + \nabla_x \,\boldsymbol{.}\, \mathbf{F}\,(u(\mathbf{x}, t)) = 0, \qquad (\mathbf{x}, t) \in \mathbb{R}^d \times [0, T]$
- $u(\mathbf{x}, 0) = u_0(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^d$

Fundamental difficulty

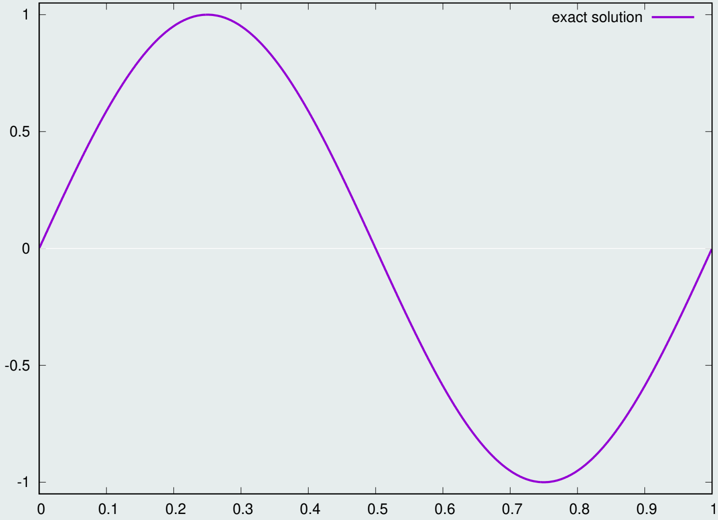
- **Even considering smooth flux function $F(\cdot)$ and initial datum $u_0(\cdot)$ ($\mathcal{C}^\infty$ for instance), solution may become discontinuous in finite time**
- Standard example: Burgers equation

$$\begin{cases} \partial_t u + \partial_x\left(\frac{1}{2}u^2\right) = 0, & (x, t) \in [0, 1] \times \mathbb{R}^+, \\ u(x, 0) = \sin(2\pi x), & x \in [0, 1]\,. \end{cases}$$

Strong solution

- Local in time existence and uniqueness of a solution $u \in \mathcal{C}^1\left(\mathbb{R}^d \times [0,\ t_c[\,\right)$

Formation of a discontinuity in finite time



Weak solution $\forall\, \psi \in \mathcal{C}_0^1(\mathbb{R}^d \times \mathbb{R}^+)$

- $\displaystyle \int\int_{\mathbb{R}^d\times\mathbb{R}^+} \Big(u\,\partial_t\psi + \boldsymbol{F}(u)\,.\,\nabla_x\psi \Big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \int_{\mathbb{R}^d} u_0(\boldsymbol{x})\,\psi(\boldsymbol{x},0)\,\mathrm{d}\boldsymbol{x} = 0$
- Non-uniqueness of a weak solution $u \in L^\infty_{loc}(\mathbb{R}^d \times \mathbb{R}^+)$

Entropy definition

- Let $u \in \mathcal{C}^1\big(\mathbb{R}^d \times [0,\, t_c[\,\big)$ be a strong solution
- η a strictly convex function is an entropy if $\exists\, \phi$ s.t.

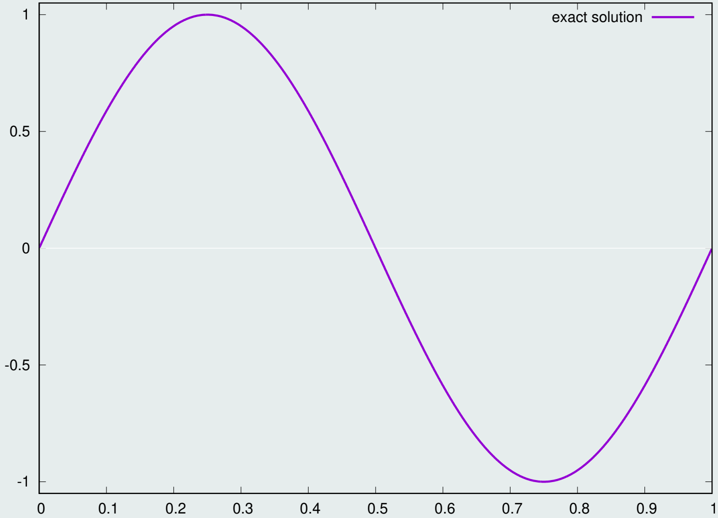
$$\partial_t\, \eta(u) + \nabla_x\,.\,\phi(u) = 0$$

Unique entropic weak solution $\forall\, \psi \in \mathcal{C}_0^1(\mathbb{R}^d \times \mathbb{R}^+)$

- Let $u \in L^\infty_{loc}(\mathbb{R}^d \times \mathbb{R}^+)$ be a weak solution
- u is the unique entropic weak solution if it satisfies, for any pair (η, ϕ)

$$\int\int_{\mathbb{R}^d\times\mathbb{R}^+} \Big(\eta(u)\,\partial_t\psi + \phi(u)\,.\,\nabla_x\psi \Big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \int_{\mathbb{R}^d} \eta\big(u_0(\boldsymbol{x})\big)\,\psi(\boldsymbol{x},0)\,\mathrm{d}\boldsymbol{x} \geq 0$$

Formation of a discontinuity in finite time



Challenges of numerical discretization

- **The weak solution we wish to approach may present areas of great regularity as well as discontinuities of great intensity**

$\longrightarrow$ **Smooth areas**: high accuracy

$\longrightarrow$ **Discontinuities**: strong robustness and stability

$\longrightarrow$ **Everywhere**: additional mathematical or physical constraints

$\hookrightarrow$ Maximum principle or positivity

$\hookrightarrow$ Entropy inequalities

$\hookrightarrow$ $\dots$

Numerical schemes and discontinuous approximated solution

- **Finite Volume** **FV**
- (Weighted) Essentially Non-Oscillatory (W)-ENO
- **Discontinuous Galerkin** **DG**

Discontinuous Galerkin scheme

- Introduced by Reed and Hill in 1973 in the frame of the neutron transport
- Major development and improvements by B. Cockburn and C.-W. Shu

Procedure

- Local variational formulation
- Piecewise polynomial approximation of the solution in the cells
- Choice of the numerical fluxes
- Time integration

Advantages

- Natural extension of Finite Volume method
- Excellent analytical properties (L_2 stability, $hp-$adaptivity, ...)
- Extremely high accuracy (superconvergent for scalar conservation laws)
- Compact stencil (involve only face neighboring cells)

Scalar conservation law

- $\partial_t\, u(\mathbf{x},t) + \nabla_x \,.\, \mathbf{F}\,(u(\mathbf{x},t)) = 0, \qquad (\mathbf{x},t) \in \omega \times [0,T]$
- $u(\mathbf{x},0) = u_0(\mathbf{x}), \qquad \mathbf{x} \in \omega$

$(k+1)^{\text{th}}$ order semi-discretization

- $\{\omega_c\}_c$ a partition of ω, such that $\omega = \bigcup_c \omega_c$
- $u_h(\mathbf{x},t)$ the numerical solution, such that $u_{h|\omega_c} = u_h^c \in \mathbb{P}^k(\omega_c)$

$$u_h^c(\mathbf{x},t) = \sum_{m=1}^{N_k} u_m^c(t)\,\sigma_m^c(\mathbf{x})$$

- $\{\sigma_m^c\}_{m=1,\ldots,N_k}$ a basis of $\mathbb{P}^k(\omega_c)$, with $N_k = \frac{(k+1)(k+2)}{2}$ in 2D.

Local variational formulation on ω_c

- $\displaystyle\int_{\omega_c} \frac{\partial\, u_h^c}{\partial t}\,\psi\,\mathrm{d}V = \oint_{\omega_c} \mathbf{F}(u_h^c)\,.\,\nabla_x\psi\,\mathrm{d}V - \oint_{\partial\omega_c} \psi\,\mathcal{F}_n\,\mathrm{d}S, \qquad \forall\,\psi \in \mathbb{P}^k(\omega_c)$
- $\mathcal{F}_n = \mathcal{F}\,(u_h^c, u_h^v, \mathbf{n})$ **numerical flux**

Numerical example: solid body rotation

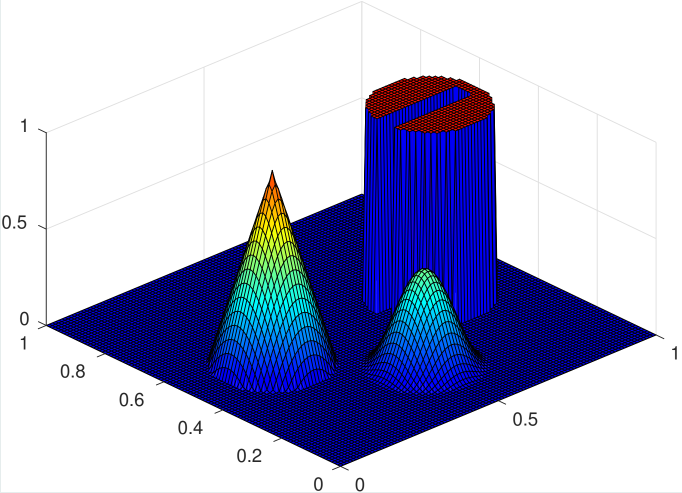
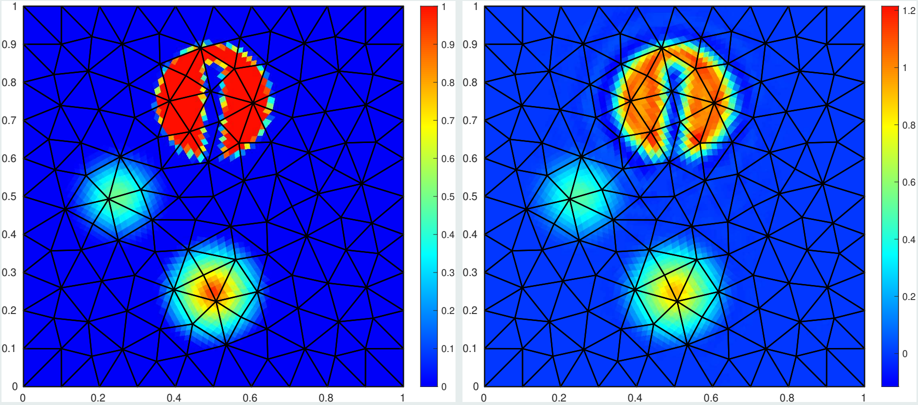


Figure: Rotation of composite signal: initial solution

Rotation of composite signal



(a) Initial solution (b) Solution after one rotation

Figure: 6th-order DG scheme on 242 cells

Rotation of composite signal

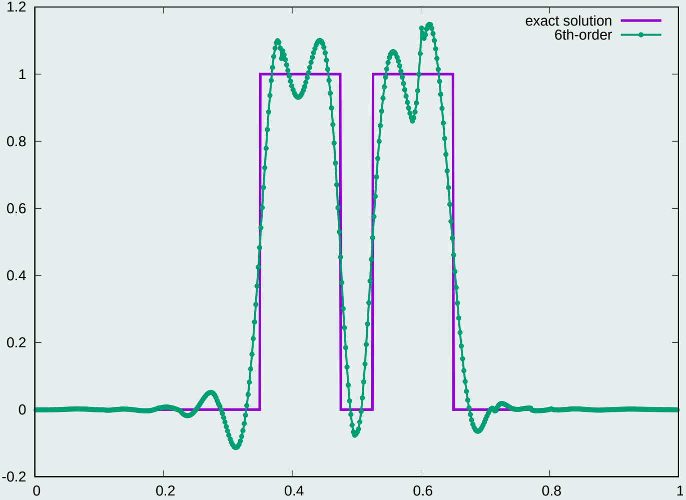


Figure: 6th-order DG scheme on 242 cells: profiles for $y = 0.75$

Spurious oscillations, aliasing and non-entropic behavior

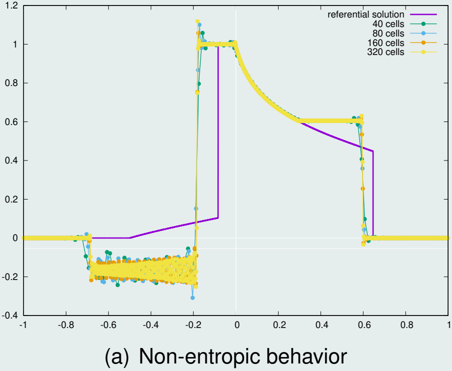


Figure : DG solutions for the Buckley non-convex flux case

Gibbs phenomenon and non-admissible solution

- High-order schemes leads to spurious oscillations near discontinuities
- Non-admissible solution potentially leading to a crash
- Vast literature of how prevent this phenomenon to happen:

$\Longrightarrow$ *a priori* and *a posteriori* limitations

A priori limitation

- Artificial viscosity
- Slope/moment/hierarchical limiter
- ENO/WENO limiter
- Flux limiter and FCT schemes
- **Monolithic HO/LO schemes**

A posteriori limitation

- MOOD ("Multi-dimensional Optimal Order Detection")
- A posteriori subcell correction

Admissible numerical solution

- Maximum principle / positivity preserving
- Limit the apparition of spurious oscillations
- Ensure a correct entropic behavior

Preserving high-accuracy and subcell resolution

Reduce the characteristic length of action

Methodology

Blend, at the subcell scale, high-order DG and 1st-order FV

F.V., *A Posteriori Correction of High-Order DG Scheme through Subcell Finite Volume Formulation and Flux Reconstruction.* JCP, 2018.

F.V., *Local subcell monolithic DG/FV convex property preserving scheme on unstructured grids and entropy consideration.* JCP, 2024.

DG as a subcell Finite Volume

- Rewrite DG scheme as a FV-like scheme on a subgrid

Cell subdivision into N_S subcells $\quad N_S \geq N_k \equiv \#\mathbb{P}^k$

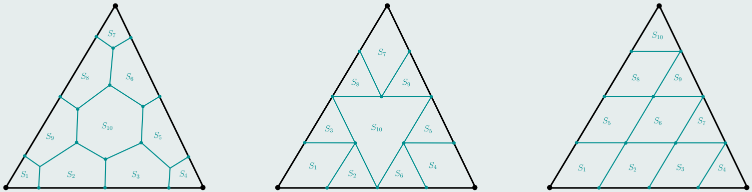


Figure : Examples of $N_s = N_k$ subdivision for $\mathbb{P}^3$ DG scheme on a triangle

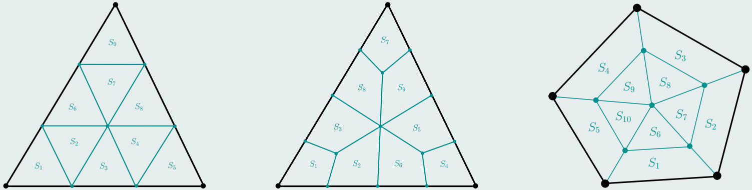


Figure : Examples of $N_S \geq N_k$ subdivision

DG schemes through residuals

- $$\sum_{m=1}^{N_k} \frac{\mathrm{d}\, u_m^c}{\mathrm{d}t} \int_{\omega_c} \sigma_m \, \sigma_p \, \mathrm{d}V = \oint_{\omega_c} \mathbf{F}(u_h^c) \,.\, \nabla_x \sigma_p \, \mathrm{d}V - \oint_{\partial\omega_c} \sigma_p \, \mathcal{F}_n \, \mathrm{d}S, \quad \forall\, p \in [\![1, N_k]\!]$$

$$\Longrightarrow \quad \boxed{M_c \, \frac{\mathrm{d}\, \mathsf{U}_c}{\mathrm{d}t} = \Phi_c}$$

- $(\mathsf{U}_c)_m = u_m^c$ — Solution moments
- $(M_c)_{mp} = \displaystyle\int_{\omega_c} \sigma_m \, \sigma_p \, \mathrm{d}V$ — Mass matrix
- $(\Phi_c)_m = \displaystyle\oint_{\omega_c} \mathbf{F}(u_h^c) \,.\, \nabla_x \sigma_m \, \mathrm{d}V - \oint_{\partial\omega_c} \sigma_m \, \mathcal{F}_n \, \mathrm{d}S$ — DG residuals

Subdivision and definition

- ω_c is subdivided into N_s subcells S_m^c
- Let us define $\overline{\psi}_m^c = \dfrac{1}{|S_m^c|} \displaystyle\int_{S_m^c} \psi \, \mathrm{d}V$ the subcell mean value

Submean values

- $\overline{u}_m^c = \dfrac{1}{|S_m^c|} \sum_{q=1}^{N_k} u_q^c \int_{S_m^c} \sigma_q \,\mathrm{d}V \quad \Longrightarrow \quad \boxed{\overline{\mathrm{U}}_c = P_c \,\mathrm{U}_c}$
- $(\overline{\mathrm{U}}_c)_m = \overline{u}_m^c$ — Submean values
- $(P_c)_{mp} = \dfrac{1}{|S_m^c|} \int_{S_m^c} \sigma_p \,\mathrm{d}V$ — Projection matrix

$$\Longrightarrow \quad \boxed{\frac{\mathrm{d}\,\overline{\mathrm{U}}_c}{\mathrm{d}t} = P_c \, M_c^{-1} \, \Phi_c}$$

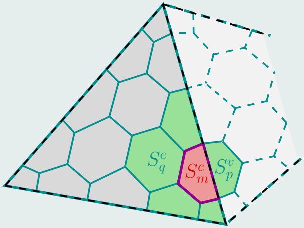
Admissibility of the cell sub-partition into subcells

- $P_c^{\mathrm{t}} \, P_c$ **has to be non-singular**

$$\Longrightarrow \quad \boxed{\mathrm{U}_c = (P_c^{\mathrm{t}} \, P_c)^{-1} P_c^{\mathrm{t}} \, \overline{\mathrm{U}}_c}$$ Least square procedure

- If $N_s = N_k$, $\quad \overline{\mathrm{U}}_c = P_c \,\mathrm{U}_c \quad \Longleftrightarrow \quad \mathrm{U}_c = P_c^{-1} \overline{\mathrm{U}}_c$

Subcell Finite Volume: reconstructed fluxes



- Let us introduce the **reconstructed fluxes**

$$\boxed{\frac{\mathrm{d}\,\overline{u}_m^c}{\mathrm{d}t} = -\frac{1}{|S_m^c|} \sum_{S_p^v \in \mathcal{V}_m^c} l_{mp}\, \widehat{F_{pm}}}$$

- $\mathcal{V}_m^c$ the set of face neighboring subcells of S_m^c
- We impose that on the boundary of cell ω_c, so for $S_p^v \notin \omega_c$

$$\boxed{l_{mp}\, \widehat{F_{pm}} = \oint_{f_{mp}^c} \mathcal{F}_n \,\mathrm{d}S \equiv \oint_{f_{mp}^c} \mathcal{F}\left(u_h^c,\, u_h^v,\, \boldsymbol{n}_{mp}^c\right)\,\mathrm{d}S}$$

- $\breve{\mathcal{V}}_m^c$ the set of face neighboring subcells of S_m^c belonging to ω_c
- Let A_c be the adjacency matrix such that

$$(A_c)_{mp} = \begin{cases} 1 & \text{if } S_p^v \in \breve{\mathcal{V}}_m^c \text{ with } m < p, \\ -1 & \text{if } S_p^v \in \breve{\mathcal{V}}_m^c \text{ with } m > p, \\ 0 & \text{if } S_p^v \notin \breve{\mathcal{V}}_m^c. \end{cases}$$

Subcell Finite Volume: reconstructed fluxes

- Let us introduce $D_c = \mathrm{diag}\left(|S_1^c|, \ldots, |S_{N_k}^c|\right)$ and $(\mathrm{B}_c)_m = \displaystyle\int_{\partial S_m^c \cap \partial\omega_c} \mathcal{F}_n \,\mathrm{d}S$
- Let $\widehat{\mathrm{F}_c}$ be the vector containing all the interior faces reconstructed fluxes

$$\boxed{-A_c\,\widehat{\mathrm{F}_c} = D_c\,P_c\,M_c^{-1}\,\Phi_c + \mathrm{B}_c}$$

Graph Laplacian technique

- $A_c \in \mathcal{M}_{N_s \times N_f^c}$ with N_f^c the number of interior faces
- $(L_c)_{mp} := \left(A_c\,A_c^{\mathrm{t}}\right)_{mp} = \begin{cases} \left|\,\widetilde{\mathcal{V}_m^c}\,\right| & \text{if } m = p, \\ -1 & \text{if } S_p^v \in \widetilde{\mathcal{V}_m^c}, \\ 0 & \text{otherwise.} \end{cases}$
- $L_c\,\mathbf{1} = \mathbf{0}$ where $\mathbf{1} = (1, \ldots, 1)^{\mathrm{t}} \in \mathbb{R}^{N_s}$
- $\Pi = \frac{1}{N_s}\,(\mathbf{1} \otimes \mathbf{1}) \in \mathcal{M}_{N_s}$

Graph Laplacian technique

- Let $\mathcal{L}_c^{-1}$ be the pseudo-inverse of L_c such that

$$\mathcal{L}_c^{-1} = (L_c + \lambda\,\Pi)^{-1} - \frac{1}{\lambda}\,\Pi \qquad \forall\,\lambda \neq 0$$

- Then, $\widehat{\mathrm{F}_c}$ is uniquely defined as following

$$\boxed{\widehat{\mathrm{F}_c} = -A_c^{\mathrm{t}}\,\mathcal{L}_c^{-1}\left(D_c\,P_c\,M_c^{-1}\,\Phi_c + \mathrm{B}_c\right)}$$

- **The only terms depending on the time are Φ_c and B_c**
- Equivalently, the polynomial solution governing equation is given by

$$\frac{\mathrm{d}\,\mathrm{U}_c}{\mathrm{d}t} = -P_c^{-1}\,D_c^{-1}\left(A_c\,\widehat{\mathrm{F}_c} + \mathrm{B}_c\right)$$

remark

- This unique solution does exit since $\left(D_c\,P_c\,M_c^{-1}\,\Phi_c + \mathrm{B}_c\right) \cdot \mathbf{1} = 0$

Different cell subdivisions $\mathbb{P}^3$

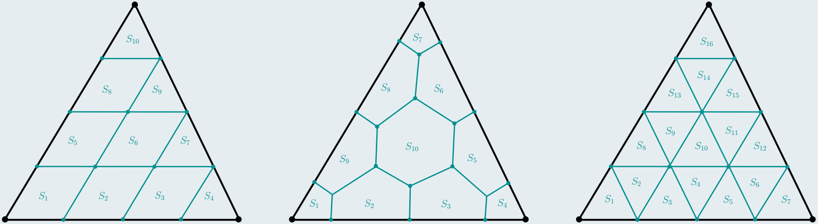
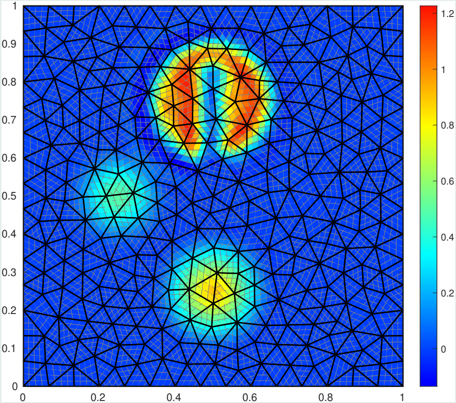


Figure: Examples of easily generalizable subdivisions for a triangle cell

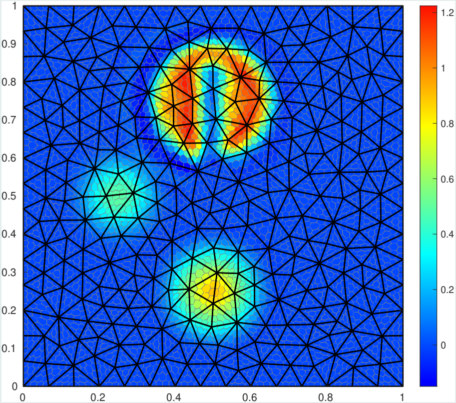
DG is DG

- **Only the functional space matters**
- **The cell subdivision has no influence on the resulting scheme**
- **Even in the case where** $\boldsymbol{N_s > N_k}$

Rotation of a composite signal after one full rotation



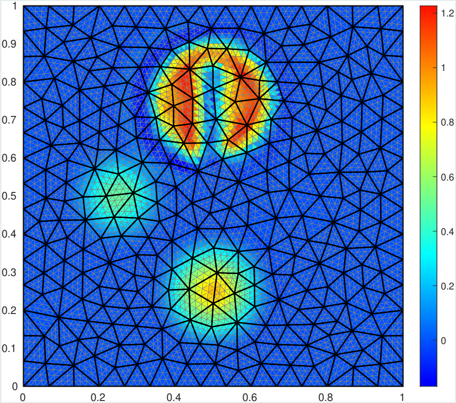
(a) Cartesian subdivision



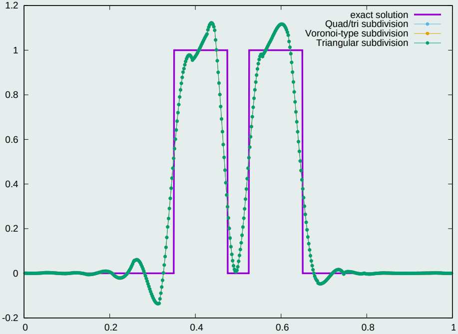
(b) Polygonal subdivision

Figure: $\mathbb{P}^3$ reconstructed flux FV schemes on 576 cells

Rotation of a composite signal after one full rotation



(c) Triangular subdivision



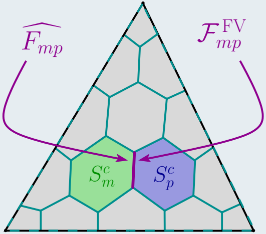
(d) Solution profiles for $y = 0.75$

Figure: $\mathbb{P}^3$ reconstructed flux FV schemes on 576 cells

Blending low and high order fluxes

- Each face f^c_{mp} of each subcell S^c_m will be assigned two fluxes

- $$\boxed{\widetilde{F}_{mp} = \mathcal{F}^{\text{FV}}_{mp} + \underbrace{\theta_{mp}}_{\in[0,1]} \underbrace{\left(\widehat{F}_{mp} - \mathcal{F}^{\text{FV}}_{mp}\right)}_{\Delta F_{mp}}}$$



- $\mathcal{F}^{\text{FV}}_{mp} := \mathcal{F}\left(\overline{u}^c_m, \overline{u}^v_p, \boldsymbol{n}_{mp}\right)$ first-order subcell numerical flux
- $\widehat{F}_{mp}$ high-order DG reconstructed flux
- **The local subcell monolithic DG/FV then writes as follows**

$$\boxed{\frac{\mathrm{d}\,\overline{u}^c_m}{\mathrm{d}t} = -\frac{1}{|S^c_m|} \sum_{S^v_p \in \mathcal{V}^c_m} l_{mp}\, \widetilde{F}_{mp}}$$

Numerical flux **E-flux**

$$\boxed{\mathcal{F}(u^-, u^+, \boldsymbol{n}) = \frac{\left(\boldsymbol{F}(u^-) + \boldsymbol{F}(u^+)\right)}{2} \cdot \boldsymbol{n} - \frac{\gamma(u^-, u^+, \boldsymbol{n})}{2}\,(u^+ - u^-)}$$

- $\gamma(u^-, u^+, \boldsymbol{n}) \geq \max\limits_{w \in \mathrm{I}(u^-, u^+)} \left(|\boldsymbol{F}'(w) \cdot \boldsymbol{n}|\right)$
- $\mathrm{I}(a, b) = [\min(a, b), \max(a, b)]$
- **In the system case, we make use of either Rusanov, HLL, HLL-C, . . .**

Time integration

- For sake of simplicity, we focus on forward Euler (FE) time stepping, as SSP Runge-Kutta can be formulated as convex combinations of FE
- The semi-discrete scheme provided with FE time integration writes

$$\boxed{\overline{u}_m^{c,n+1} = \overline{u}_m^{c,n} - \frac{\Delta t}{|S_m^c|} \sum_{S_p^v \in \mathcal{V}_m^c} l_{mp}\, \widetilde{F_{mp}}}$$

Reformulation of the monolithic subcell scheme

- $\gamma_{mp} := \gamma\left(\overline{u}_m^{c,n}, \overline{u}_p^{v,n}, \boldsymbol{n}_{mp}\right)$ 1st-order FV dissipation coefficient

- $$\overline{u}_m^{c,n+1} = \overline{u}_m^{c,n} - \frac{\Delta t}{|S_m^c|}\left(\sum_{S_p^v \in \mathcal{V}_m^c} l_{mp}\,\widetilde{F_{mp}} \pm \sum_{S_p^v \in \mathcal{V}_m^c} l_{mp}\,\gamma_{mp}\,\overline{u}_m^{c,n} + \boldsymbol{F}(\overline{u}_m^{c,n}) \boldsymbol{.} \sum_{S_p^v \in \mathcal{V}_m^c} l_{mp}\,\boldsymbol{n}_{mp}\right)$$

$$= \left(1 - \frac{\Delta t}{|S_m^c|}\sum_{S_p^v \in \mathcal{V}_m^c} l_{mp}\,\gamma_{mp}\right)\overline{u}_m^{c,n}$$

$$+ \frac{\Delta t}{|S_m^c|}\sum_{S_p^v \in \mathcal{V}_m^c} l_{mp}\,\gamma_{mp}\left(\overline{u}_m^{c,n} - \frac{\widetilde{F_{mp}} - \boldsymbol{F}(\overline{u}_m^{c,n}) \boldsymbol{.}\, \boldsymbol{n}_{mp}}{\gamma_{mp}}\right)$$

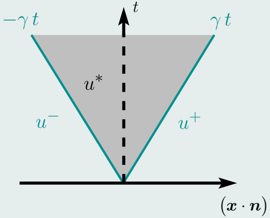
- $$\boxed{\overline{u}_m^{c,n+1} = \left(1 - \frac{\Delta t}{|S_m^c|}\sum_{S_p^v \in \mathcal{V}_m^c} l_{mp}\,\gamma_{mp}\right)\overline{u}_m^{c,n} + \frac{\Delta t}{|S_m^c|}\sum_{S_p^v \in \mathcal{V}_m^c} l_{mp}\,\gamma_{mp}\,\widetilde{\overline{u}_{mp}}}$$

- **Convex combination under CFL condition**

Blended Riemann intermediate states

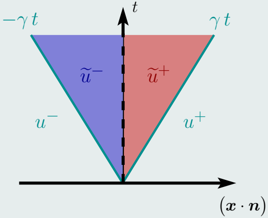
- $$\widetilde{u_{mp}}^{-} = \overline{u}_m^{c,n} - \frac{\mathcal{F}_{mp}^{\mathrm{FV}} - \boldsymbol{F}(\overline{u}_m^{c,n}) \cdot \boldsymbol{n}_{mp}}{\gamma_{mp}} - \theta_{mp}\frac{\Delta F_{mp}}{\gamma_{mp}}$$
$$= \underbrace{\frac{\overline{u}_m^{c,n} + \overline{u}_p^{v,n}}{2} - \frac{\left(\boldsymbol{F}(\overline{u}_p^{v,n}) - \boldsymbol{F}(\overline{u}_m^{c,n})\right) \cdot \boldsymbol{n}_{mp}}{2\,\gamma_{mp}}}_{u_{mp}^{*,\,\mathrm{FV}}} - \theta_{mp}\frac{\Delta F_{mp}}{\gamma_{mp}}$$

- $$\widetilde{u_{mp}}^{\pm} = u_{mp}^{*,\,\mathrm{FV}} \pm \theta_{mp}\frac{\Delta F_{mp}}{\gamma_{mp}} \quad \Longrightarrow \quad u_{mp}^{*,\,\mathrm{FV}} = \frac{1}{2}\left(\widetilde{u_{mp}}^{-} + \widetilde{u_{mp}}^{+}\right)$$



(a) 1st-order situation

$\Longrightarrow$



(b) Blended flux situation

Admissible solution — Find the correct θ_{mp}

- G a convex admissible set where the solution has to remain in
  - $G = \left\{ u_0 \in [\alpha, \beta] \implies u \in [\alpha, \beta] \right\}$ SCL
  - $G = \left\{ u = \begin{pmatrix} \rho \\ \mathbf{q} \\ E \end{pmatrix}, \quad \rho > 0,\, p(u) > 0 \right\}$ Euler
- **A sufficient condition to ensure $\overline{u}_m^{c,n+1} \in G$ is that**

$$\boxed{\forall\, S_p^v \in \mathcal{V}_m^c, \qquad \widetilde{u_{mp}}^{\pm} = u_{mp}^{*,\,\mathrm{FV}} \pm \theta_{mp} \frac{\Delta F_{mp}}{\gamma_{mp}} \in G}$$

- We want to prevent to code from crashing (apparition of NaN, . . .)
- We want to prevent the apparition of spurious oscillations
- ***We want to ensure discrete entropy inequalities (?)***
- We apply a local blending coefficients smoothening to avoid too stiff transition from high to low orders

Questions regarding entropy

- **Can we find the θ_{mp} coefficients ensuring an entropy inequality?**
- **What do we mean by entropy inequality?**
  - for one or any entropy?
  - at the discrete or semi-discrete time level?
  - at the cells or subcells space level?
- **If we manage to ensure an entropy inequality, is it worth the effort?**
  - in terms of accuracy
  - in terms of other critical properties to ensure, as positivity for instance
- **Do we really need an entropy inequality to practically capture the entropic weak solution?**

Definitions

- (η, ϕ) entropy - entropy flux
- $v(u) = \eta'(u)$ entropy variable
- $\psi(u) = v(u)\, \boldsymbol{F}(u) - \phi(u)$ entropy potential flux

Subcell entropy stability at the discrete level **for all** (η, ϕ)

- if $\Delta F_{mp} \,.\, \left(\overline{u}_p^{v,n} - \overline{u}_m^{c,n}\right) > 0$,

$$\boxed{\theta_{mp} \leq \min\left(1,\ \frac{\left(\gamma_{mp} - \gamma_{\max}\right)\left(\overline{u}_p^{v,n} - \overline{u}_m^{c,n}\right)}{2\,\Delta F_{mp}}\right)}$$

- $\gamma_{\max} \quad := \max\limits_{w \,\in\, \mathrm{I}\,(\overline{u}_m^{c,n}, \overline{u}_p^{v,n})} \left(\left|\boldsymbol{F}'(w) \,.\, \boldsymbol{n}_{mp}\right|\right)$

$\Longrightarrow$ <u>**1st order scheme!**</u>

Semi-discrete subcell entropy dissipation **for a given** (η, ϕ)

- if $\Delta F_{mp} \cdot \left(v(\overline{u}_p^{v,n}) - v(\overline{u}_m^{c,n}) \right) > 0,$

$$\boxed{\theta_{mp} \leq \min\left(1, \frac{\left(\dfrac{\psi(\overline{u}_p^{v,n}) - \psi(\overline{u}_m^{c,n})}{v(\overline{u}_p^{v,n}) - v(\overline{u}_m^{c,n})}\right) \cdot \boldsymbol{n}_{mp} - \mathcal{F}_{mp}^{\mathrm{FV}}}{\Delta F_{mp}}\right)}$$

$\Longrightarrow$ **2nd order scheme!**

A. Rueda-Ramírez, B. Bolm, D. Kuzmin and **G. Gassner**, *Monolithic Convex Limiting for Legendre-Gauss-Lobatto Discontinuous Galerkin Spectral Element Methods.* Commun. Appl. Math. Comput., 2024.

Sub-resolution basis functions $N_s = N_k$

- Let $\{\varphi_m^c\}_m$ be the sub-resolution basis function such that, $\forall \psi \in \mathbb{P}^k(\omega_c)$

$$\int_{\omega_c} \varphi_m \, \psi \, \mathrm{d}V = \int_{S_m^c} \psi \, \mathrm{d}V$$

- Then, given $v_h^c \in \mathbb{P}^k(\omega_c)$, it writes

$$v_h^c = \sum_{m=1}^{N_k} \underline{v}_m^c \, \varphi_m^c$$

Semi-discrete cell entropy dissipation **for a given** (η, ϕ)

- $\Delta\eta_c := \dfrac{\mathrm{d}}{\mathrm{d}t} \displaystyle\oint_{\omega_c} \eta(u_h^c) \, \mathrm{d}V = \oint_{\omega_c} v(u_h^c) \, \partial_t u_h^c \, \mathrm{d}V = \int_{\omega_c} v_h^c \, \partial_t u_h^c \, \mathrm{d}V$

- $v_h^c = \displaystyle\sum_{m=1}^{N_k} \underline{v}_m^c \, \varphi_m^c$ $\quad$ L^2 projection of $v(u_h^c)$ onto $\mathbb{P}^k$

- $\Delta\eta_c = \displaystyle\sum_{m=1}^{N_k} \underline{v}_m^c \int_{\omega_c} \varphi_m^c \, \partial_t u_h^c \, \mathrm{d}V = \sum_{m=1}^{N_k} |S_m^c| \, \underline{v}_m^c \, \dfrac{\mathrm{d}\,\overline{u}_m^c}{\mathrm{d}t} = -\sum_{m=1}^{N_k} \underline{v}_m^c \sum_{S_p^v \in \mathcal{V}_m^c} l_{mp} \, \widetilde{F_{mp}}$

Continuous Knapsack problem

- The sufficient condition rewrites as $\boxed{\mathbf{C}_c \cdot \boldsymbol{\Theta}_c \leq \mathrm{D}_c}$

- $\boldsymbol{\Theta}$ contains all the subcells' faces, $\mathbf{C}_c$ the vector defined as

$$\mathrm{C}^c_{mp} = \begin{cases} \displaystyle\oint_{f_{mp}} \Big(\big(v(u^c_h) - \underline{v}^c_m \big)\, \mathcal{F}_n - \Big(\psi(u^c_h) - \psi\big(u(\underline{v}^c_m)\big) \Big) \cdot \boldsymbol{n}_{mp} \Big) \,\mathrm{d}S, & \forall f_{mp} \subset \partial\omega_c \\ l_{mp}\, \big(\underline{v}^c_p - \underline{v}^c_m\big)\, \Delta F_{mp}, & \text{otherwise} \end{cases}$$

- $\mathrm{D}_c = \displaystyle\sum_{f_{mp} \in \hat{\mathfrak{f}}_c} l_{mp}\, \psi\big(u(\underline{v}^c_m)\big) \cdot \boldsymbol{n}_{mp} - \sum_{f_{mp} \in \check{\mathfrak{f}}_c} l_{mp}\, \big(\underline{v}^c_p - \underline{v}^c_m\big)\, \mathcal{F}^{\mathrm{FV}}_{mp}$

- $\mathcal{F}^{\mathrm{FV}}_{mp} = \mathcal{F}\Big(u\big(\underline{v}^c_m\big),\, u\big(\underline{v}^v_p\big),\, \boldsymbol{n}_{mp} \Big)$ **modified FV numerical flux**

- **Because** $\mathrm{D}_c \geq 0$ **the Knapsack problem is indeed solvable**

Y. LIN AND J. CHAN, *High order entropy stable discontinuous Galerkin spectral element methods through subcell limiting.* JCP, 2024.

Greedy algorithm

- Find $\mathbf{0} \leq \boldsymbol{\Theta} \leq \boldsymbol{\Theta}^e \leq \mathbf{1}$ maximizing $\sum \theta_{mp}$ such that $\boxed{\mathbf{C}_c \,.\, \boldsymbol{\Theta}_c \leq \mathrm{D}_c}$
- θ_{mp}^e being any given supplementary constraint

High-order accuracy preservation

- Substituting u_h by a smooth solution u
- h_c is the diameter of cell ω_c
- Then, we have that

$$\boxed{\mathbf{C}_c \,.\, \mathbf{1} - \mathrm{D}_c = |\omega_c| \, \mathcal{O}(h_c^{k+1})}$$

- This finally implies that

$$\boxed{\widetilde{F_{mp}} = \widehat{F_{mp}} + (\theta_{mp} - 1) \, \Delta F_{mp} = \widehat{F_{mp}} + \mathcal{O}(h_c^{k+2})}$$

Linear advection of a sinus $\eta(u) = \frac{1}{2}u^2$

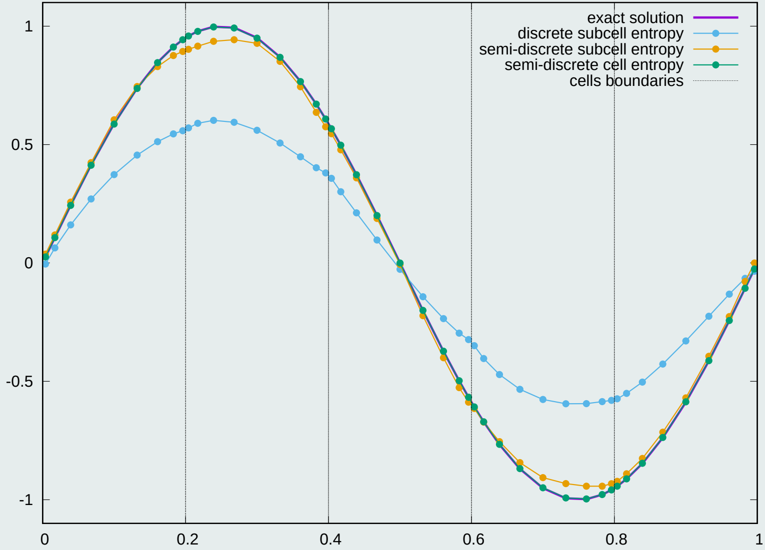


Figure: $\mathbb{P}^8$-DG/FV with entropy stabilities on 5 cells

Linear advection of a composite signal $\eta(u) = \frac{1}{2}u^2$

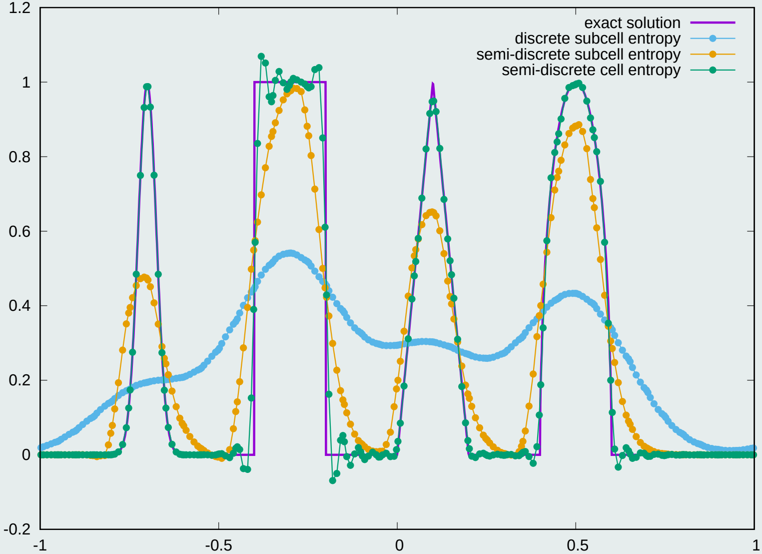
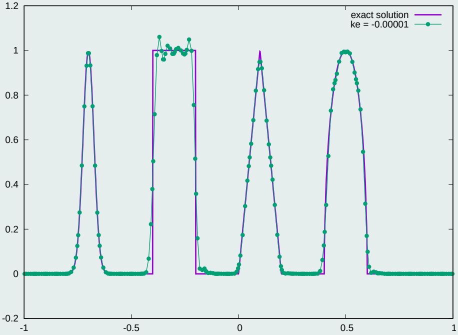
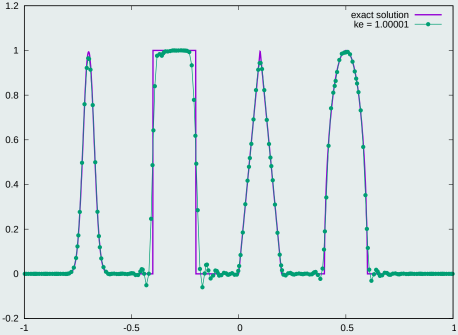


Figure: $\mathbb{P}^5$-DG/FV with entropy stabilities on 40 cells

Linear advection of a composite signal $\eta(u) = |u - k_e|^{1+\epsilon} \backslash (1+\epsilon)$



(a) $k_e = -0.00001$



(b) $k_e = 1.00001$

Figure: $\mathbb{P}^5$-DG/FV submean values on 40 cells: $\epsilon = 0.25$

Linear advection of a composite signal $\eta(u) = |u - k_e|^{1+\epsilon} \backslash (1+\epsilon)$

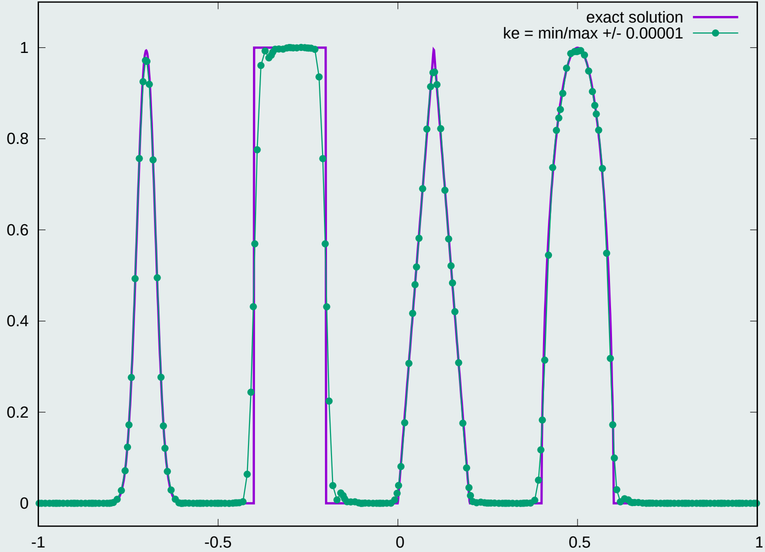


Figure: $\mathbb{P}^5$-DG/FV on 40 cells: $k_e = -0.00001$ and $k_e = 1.00001$

Non-linear non-convex flux Buckley case 80 cells

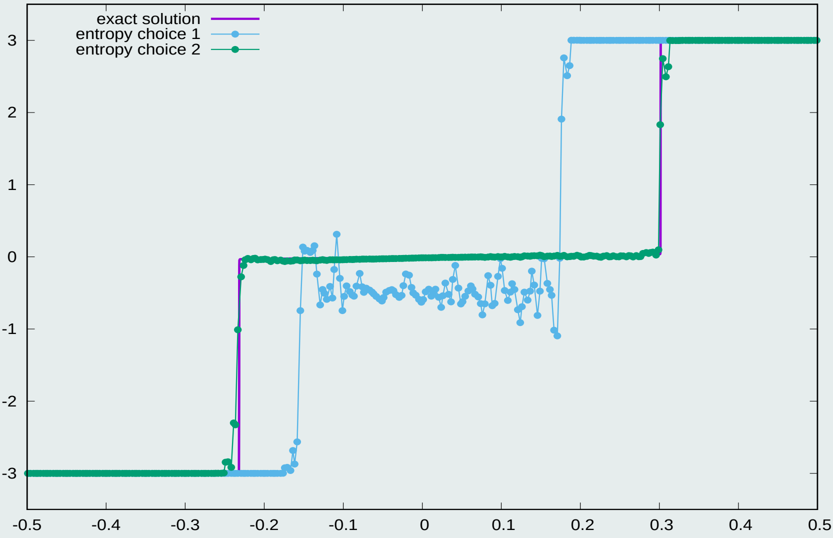


Figure: $\mathbb{P}^3$-DG/FV submean values: $\eta_1(u) = \frac{1}{2}u^2$ and $\eta_2(u) = \int \mathrm{atan}(20u)\,\mathrm{d}u$

Non-linear non-convex flux Buckley case 80 cells

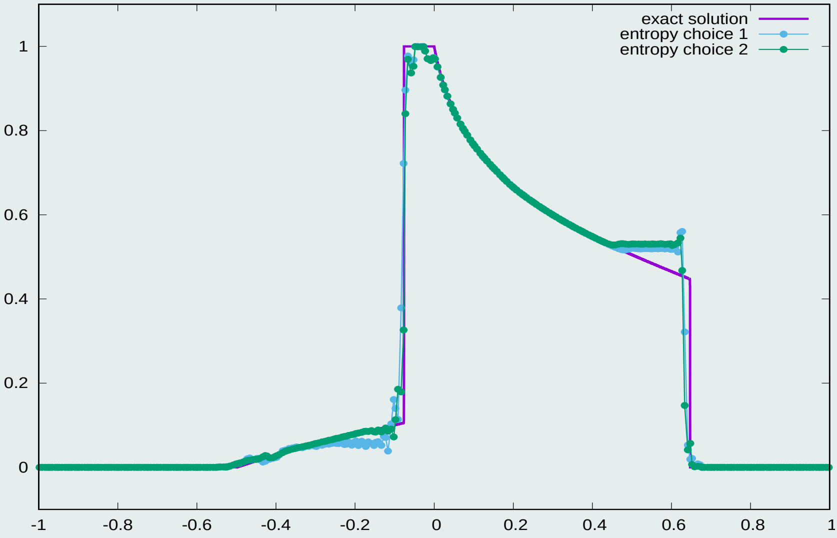
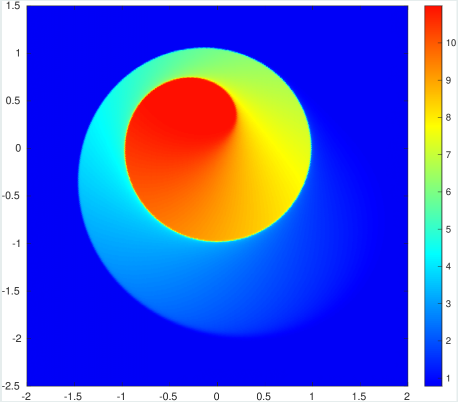
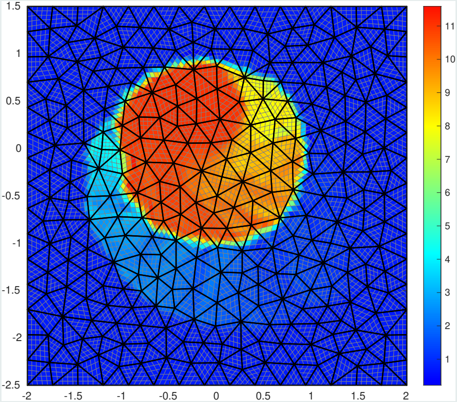


Figure: $\mathbb{P}^3$-DG/FV submean values: $\eta_1(u) = \frac{1}{2}u^2$ and $\eta_2(u) = \int \operatorname{atan}(20u)\,\mathrm{d}u$

KPP non-convex flux problem $\eta(u) = \frac{1}{2}u^2$



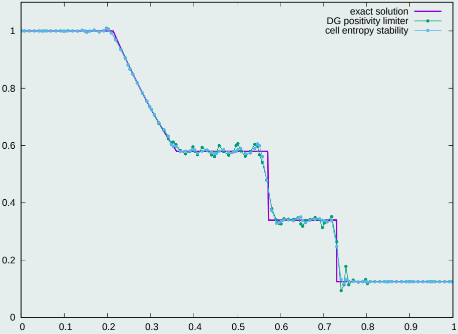
(a) 1^{th}-order FV on 209184 cells



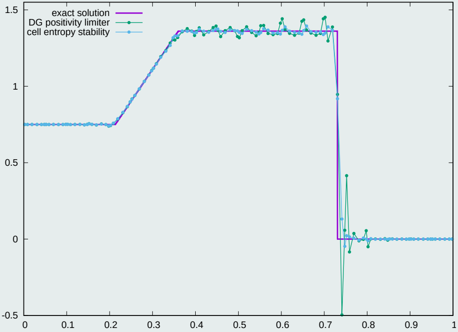
(b) $\mathbb{P}^4$-DG/FV on 576 cells

Figure: $\mathbb{P}^4$-DG/FV entropic scheme: non-entropic solution

1D modified Sod shock tube problem $\eta(U) = -\rho \log\left(p/\rho^\gamma\right)$



(a) Density



(b) Velocity

Figure: $\mathbb{P}^5$ pure DG with positivity limiter and $\mathbb{P}^5$-DG/FV monolithic scheme with cell entropy stability on 20 cells

Questions regarding entropy $\longrightarrow$ pieces of answer

- **Can we find θ_{mp} coefficients ensuring an entropy inequality?**

  $\hookrightarrow$ **Yes!**

- **What do we mean by entropy inequality, and is it worth the effort?**
  - for any entropy, at the discrete time level and for any subcell

    $\hookrightarrow$ **1st-order**
  - for a given entropy, at the semi-discrete time level for any subcells

    $\hookrightarrow$ **2nd-order**
  - for a given entropy, at the semi-discrete time level for any cells

    $\hookrightarrow$ $(k+1)^{\text{th}}$**-order**

    $\hookrightarrow$ $\mathcal{F}_{mp}^{\text{FV}} = \mathcal{F}\Big(u(\underline{v}_m^c),\, u(\underline{v}_p^v),\, n_{mp}\Big)$ ⚠

- **Do we absolutely need entropy stability while aiming for high order?**

  $\hookrightarrow$ **Unclear...** $\Longrightarrow$ **GMP and LMP + relaxation**

Global maximum principle $\overline{u}_m^{c,n+1} \in [\alpha,\, \beta]$

$$\boxed{\theta_{mp} \leq \min\left(1,\, \left|\frac{\gamma_{mp}}{\Delta F_{mp}}\right| \min\left(\beta - u_{mp}^{*,\,\mathrm{FV}},\, u_{mp}^{*,\,\mathrm{FV}} - \alpha\right)\right)}$$

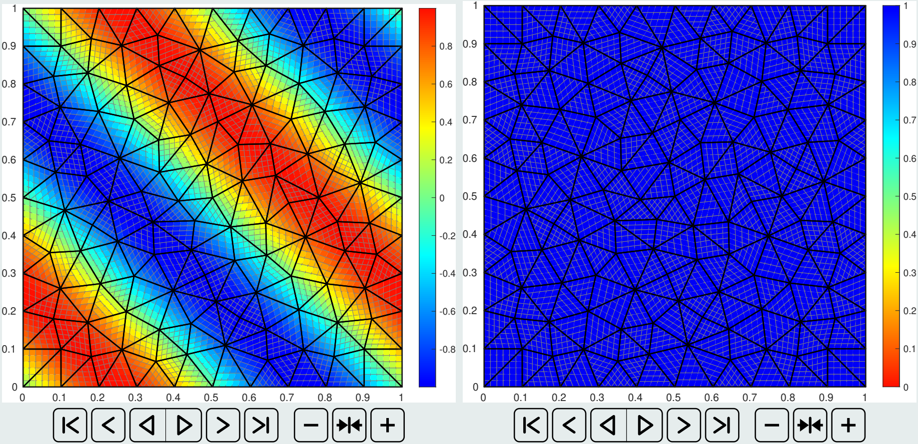
Local maximum principle $\overline{u}_m^{c,n+1} \in [\alpha_m^c,\, \beta_m^c]$

- $\alpha_m^c := \min\limits_{S_q^w \in \mathcal{N}(S_m^c)} \left(\overline{u}_q^{w,n}\right)$ and $\beta_m^c := \max\limits_{S_q^w \in \mathcal{N}(S_m^c)} \left(\overline{u}_q^{w,n}\right)$

$$\boxed{\theta_{mp} \leq \min\left(1,\, \left|\frac{\gamma_{mp}}{\Delta F_{mp}}\right| \begin{cases} \min\left(\beta_p^v - u_{mp}^{*,\,\mathrm{FV}},\, u_{mp}^{*,\,\mathrm{FV}} - \alpha_m^c\right) & \text{if } \Delta F_{mp} > 0 \\ \min\left(\beta_m^c - u_{mp}^{*,\,\mathrm{FV}},\, u_{mp}^{*,\,\mathrm{FV}} - \alpha_p^v\right) & \text{if } \Delta F_{mp} < 0 \end{cases}\right)}$$

- The wider set $\mathcal{N}(S_m^c)$ is, the softer this local maximum principle will be
- **Smooth extrema relaxation to preserve accuracy**

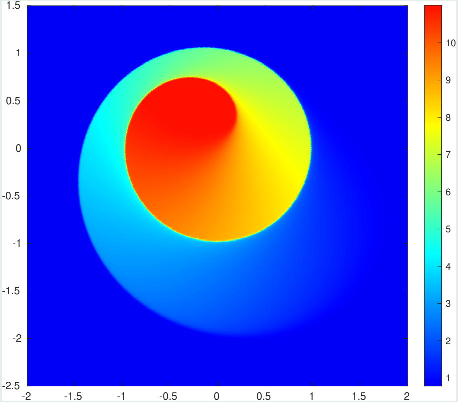
Burgers equation $u_0(x,y) = \sin(2\pi\,(x+y))$



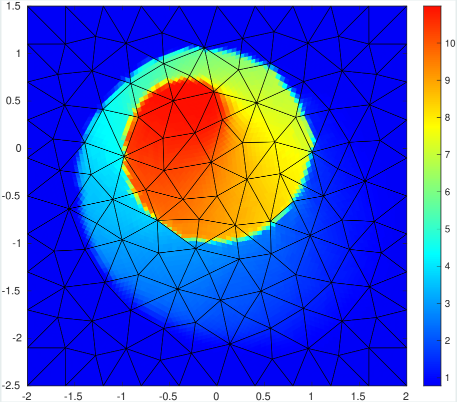
(a) Solution submean values (b) Blending coefficients

Figure: $\mathbb{P}^5$-DG/FV scheme with GMP and relaxed-LMP on 242 cells

KPP non-convex flux problem



(a) 1^{th}-order FV on 209184 cells



(b) $\mathbb{P}^7$-DG/FV on 272 cells

Figure: $\mathbb{P}^7$-DG/FV scheme with GMP and relaxed-LMP

Non-linear Euler compressible gas dynamics equations

- $\partial_t \boldsymbol{V} + \nabla_x \boldsymbol{.} \boldsymbol{F}(\boldsymbol{V}) = \boldsymbol{0}$
- $\boldsymbol{V} = \begin{pmatrix} \rho \\ \boldsymbol{q} \\ E \end{pmatrix}$ — conservative variables
- $\boldsymbol{F}(\boldsymbol{V}) = \begin{pmatrix} \boldsymbol{q} \\ \dfrac{\boldsymbol{q} \otimes \boldsymbol{q}}{\rho} + p\, I_d \\ (E+p)\, \dfrac{\boldsymbol{q}}{\rho} \end{pmatrix}$ — flux function
- $p := p(\boldsymbol{V}) = (\gamma - 1) \left(E - \dfrac{1}{2} \dfrac{\|\boldsymbol{q}\|^2}{\rho} \right)$ — equation of state

Monolithic subcell DG/FV scheme property

- Positivity of the density and internal energy, at the subcell scale

Definitions

- $\widetilde{\boldsymbol{F}_{mp}} := \boldsymbol{\mathcal{F}}_{mp}^{\mathrm{FV}} + \Theta_{mp} \underbrace{\left(\widehat{\boldsymbol{F}_{mp}} - \boldsymbol{\mathcal{F}}_{mp}^{\mathrm{FV}} \right)}_{\Delta \boldsymbol{F}_{mp}}$ convex blended flux

- $\Theta_{mp} = \begin{pmatrix} \theta_{mp}^{\rho} & \mathbf{0} & 0 \\ \mathbf{0} & \boldsymbol{\theta}_{mp}^{\boldsymbol{q}} & \mathbf{0} \\ 0 & \mathbf{0} & \theta_{mp}^{E} \end{pmatrix}$

- $\boldsymbol{\mathcal{F}}_{mp}^{\mathrm{FV}} := \boldsymbol{\mathcal{F}} \left(\overline{\boldsymbol{V}}_{m}^{c,n}, \overline{\boldsymbol{V}}_{p}^{v,n}, \boldsymbol{n}_{mp} \right)$ Global L-F, Rusanov, HLL(C), ...

Positivity of the density

- $\theta_{mp}^{\rho} = \theta_{mp}^{(1)}\, \theta_{mp}^{(2)}$

$$\boxed{\theta_{mp}^{(1)} \leq \min \left(1, \left| \frac{\gamma_{mp}}{\Delta F_{mp}^{\rho}} \right| \rho_{mp}^{*,\,\mathrm{FV}} \right)}$$

Positivity of the internal energy

- $A_{mp} = \dfrac{1}{(\gamma_{mp})^2} \left(\dfrac{1}{2} \left\| \Delta \boldsymbol{F}^{\boldsymbol{q}}_{mp} \right\|^2 - \theta^{(1)}_{mp} \, \Delta F^{\rho}_{mp} \, \Delta F^{E}_{mp} \right)$
- $B_{mp} = \dfrac{1}{\gamma_{mp}} \left(\boldsymbol{q}^{*,\,\mathrm{FV}}_{mp} \cdot \Delta \boldsymbol{F}^{\boldsymbol{q}}_{mp} - \rho^{*,\,\mathrm{FV}}_{mp} \, \Delta F^{E}_{mp} - \theta^{(1)}_{mp} \, E^{*,\,\mathrm{FV}}_{mp} \, \Delta F^{\rho}_{mp} \right)$
- $M_{mp} = \rho^{*,\,\mathrm{FV}}_{mp} \, E^{*,\,\mathrm{FV}}_{mp} - \frac{1}{2} \left\| \boldsymbol{q}^{*,\,\mathrm{FV}}_{mp} \right\|^2$

$$\boxed{\theta^{(2)}_{mp} \leq \min \left(1, \, \frac{M_{mp}}{\left| B_{mp} \right| + \max\left(0, A_{mp}\right)} \right)}$$

- $\theta^{\rho}_{mp} = \theta^{\rho\,(1)}_{mp} \, \theta^{(2)}_{mp}, \quad \theta^{q^x}_{mp} = \theta^{(2)}_{mp}, \quad \theta^{q^y}_{mp} = \theta^{(2)}_{mp}, \quad \theta^{E}_{mp} = \theta^{(2)}_{mp}$

A. Rueda-Ramírez, B. Bolm, D. Kuzmin and **G. Gassner**, *Monolithic Convex Limiting for Legendre-Gauss-Lobatto Discontinuous Galerkin Spectral Element Methods.* Commun. Appl. Math. Comput., 2024.

LMP $\quad \overline{v}_m^{c,n+1} \in [\alpha_m^c, \beta_m^c]$

- $v \in \left\{\rho,\, q^x,\, q^y,\, E\right\}$ conservative variable
- $\alpha_m^c := \min\limits_{S_q^w \in \mathcal{N}(S_m^c)} \left(\overline{v}_q^{w,n},\, v_{mq}^{*,\,\mathrm{FV}}\right)$ and $\beta_m^c := \max\limits_{S_q^w \in \mathcal{N}(S_m^c)} \left(\overline{u}_q^{w,n},\, v_{mq}^{*,\,\mathrm{FV}}\right)$

$$\boxed{\theta_{mp} \le \min\left(1,\, \left|\frac{\gamma_{mp}}{\Delta F_{mp}}\right| \begin{cases} \min\left(\beta_p^v - u_{mp}^{*,\,\mathrm{FV}},\, u_{mp}^{*,\,\mathrm{FV}} - \alpha_m^c\right) & \text{if } \Delta F_{mp} > 0 \\ \min\left(\beta_m^c - u_{mp}^{*,\,\mathrm{FV}},\, u_{mp}^{*,\,\mathrm{FV}} - \alpha_p^v\right) & \text{if } \Delta F_{mp} < 0 \end{cases}\right)}$$

- **Smooth extrema relaxation to preserve accuracy**

Sod shock tube test case — 10 cells

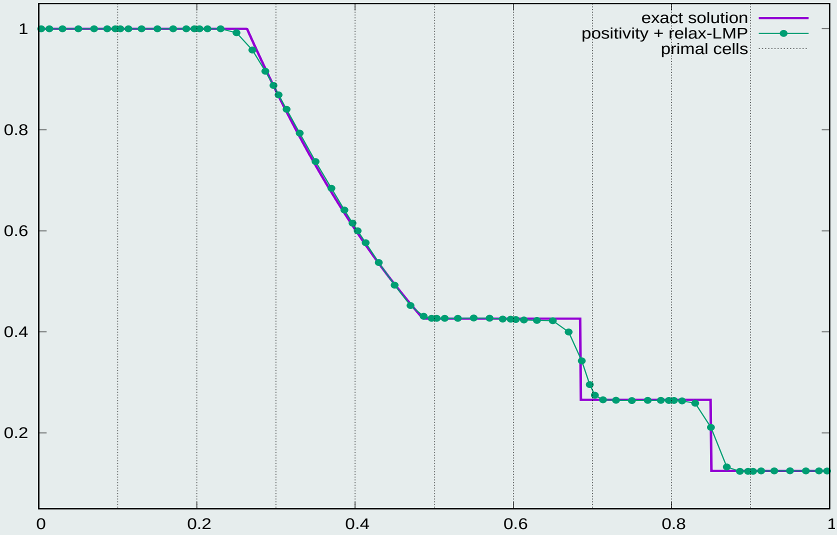
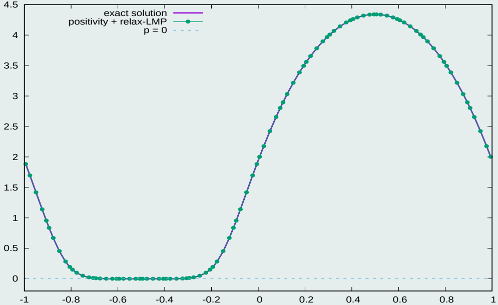


Figure: $\mathbb{P}^6$-DG/FV scheme with GMP and relaxed-LMP: submean values

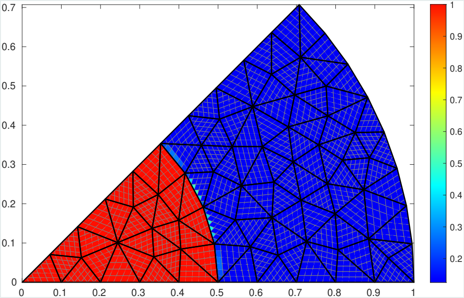
Smooth isentropic solution $\rho_0 = 1 + 0.9999999\sin(2\pi x)$



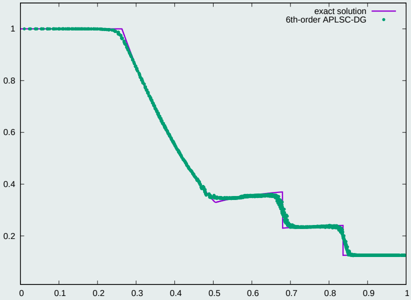
	L_1		L_2		θ_{mp}	
h	$E^h_{L_1}$	$q^h_{L_1}$	$E^h_{L_2}$	$q^h_{L_2}$	min. θ_{mp}	aver. θ_{mp}
$\frac{1}{10}$	9.07E-4	5.86	1.23E-3	5.90	2.00E-1	0.981
$\frac{1}{20}$	1.56E-5	4.03	2.05E-5	3.83	1.92E-1	0.997
$\frac{1}{40}$	9.53E-7	4.89	1.44E-6	4.85	5.65E-4	0.999
$\frac{1}{80}$	3.21E-8	4.80	5.00E-8	4.87	3.48E-5	0.999
$\frac{1}{160}$	1.15E-9	-	1.71E-9	-	1.00	1.00

Table: Convergence rates of the $\mathbb{P}^4$-DG/FV scheme with positivity and relaxed-LMP

Sod shock tube problem in cylindrical geometry



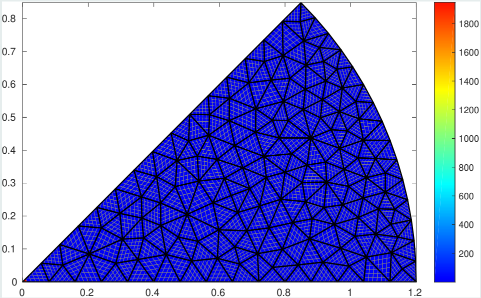
(a) Density map



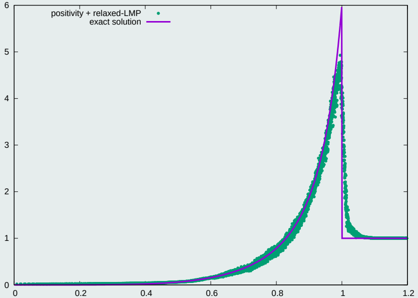
(b) Density profile

Figure: $\mathbb{P}^5$-DG/FV with positivity and relaxed-LMP on a 110 cells mesh

Sedov point blast problem in cylindrical geometry



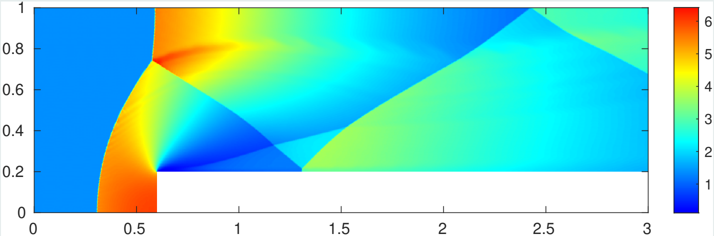
(a) Energy map



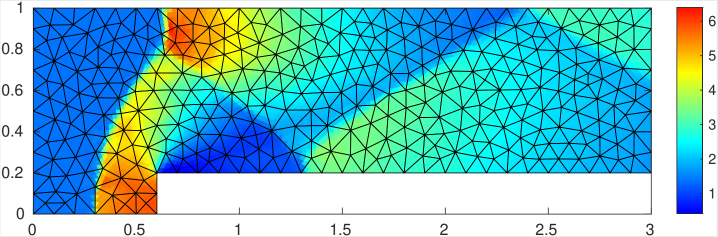
(b) Density profile

Figure: $\mathbb{P}^5$-DG/FV with positivity and relaxed-LMP on a 271 cells mesh

Mach 3 forward-facing step



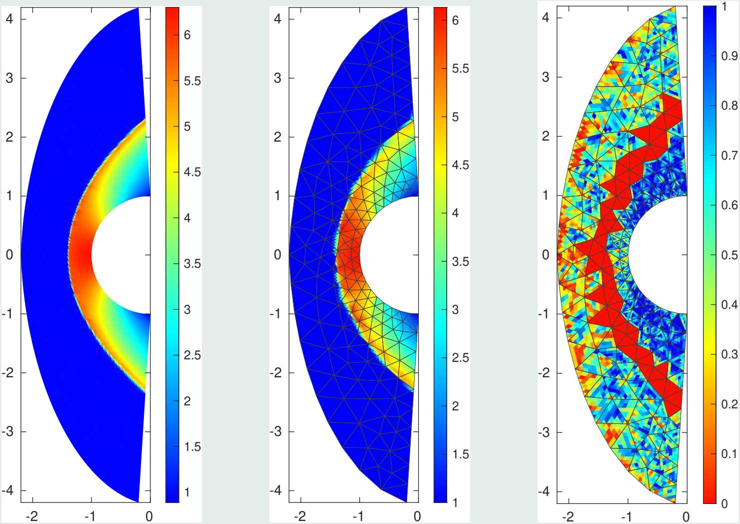
(a) $\mathbb{P}^1$ on 84108 cells



(b) $\mathbb{P}^3$ on 680 cells

Figure: Monolithic subcell DG/FV scheme with positivity and relaxed-LMP

Mach 20 hypersonic flow over half cylinder



(a) $\mathbb{P}^0$ on 25266 cells (b) $\mathbb{P}^5$ on 292 cells (c) $\mathbb{P}^5$ on 292 cells

Figure: Monolithic subcell DG/FV scheme: density and blending coefficients

Monolithic local subcell DG/FV scheme

- Reformulate DG schemes as subgrid FV-like schemes:
  - regardless the type of mesh used
  - regardless the space dimension (***in theory***...)
  - regardless the cell subdivision ($N_s \geq N_k$)
- Combine high-order reconstructed fluxes and 1^{st}-order FV fluxes
  - ensuring a maximum or positivity preserving principle at the subcell scale
  - ensuring different entropy stability inequalities
  - reducing significantly the apparition of spurious oscillations
  - preserving the very accurate subcell resolution of DG schemes

Questions

- Is an entropy inequality for one entropy enough?

  $\hookrightarrow$ **Generally, no**
- Is entropy inequality absolutely needed?

  $\hookrightarrow$ **Maybe not** $\Longrightarrow$ **GMP + relaxed LMP**

Articles on this topic

- **F.V**, *A Posteriori Correction of High-Order DG Scheme through Subcell Finite Volume Formulation and Flux Reconstruction.* JCP, 2018.
- **A. Haidar**, **F. Marche** and **F.V**, *A posteriori Finite-Volume local subcell correction of high-order discontinuous Galerkin schemes for the nonlinear shallow-water equations*. JCP, 2022.
- **F.V** and **R. Abgrall**, *A posteriori local subcell correction of DG schemes through Finite Volume reformulation on unstructured grids.* SIAM SISC, 2023.
- **A. Haidar**, **F. Marche** and **F.V**, *Free-boundary problems for wave-structure interactions in shallow-water: DG-ALE description and local subcell correction*. JSC, 2023.
- **A. Haidar**, **F. Marche** and **F.V**, *A robust DG-ALE formulation for nonlinear shallow-water interactions with a floating object*. JSC, 2024.
- **F.V**, *Monolithic local subcell DG/FV convex property preserving scheme on unstructured grids and entropy consideration.* JCP, 2024.

Articles on this topic

- **F.V**, *A Posteriori Correction of High-Order DG Scheme through Subcell Finite Volume Formulation and Flux Reconstruction.* JCP, 2018.
- **A. Haidar**, **F. Marche** and **F.V**, *A posteriori Finite-Volume local subcell correction of high-order discontinuous Galerkin schemes for the nonlinear shallow-water equations*. JCP, 2022.
- **F.V** and **R. Abgrall**, *A posteriori local subcell correction of DG schemes through Finite Volume reformulation on unstructured grids.* SIAM SISC, 2023.
- **A. Haidar**, **F. Marche** and **F.V**, *Free-boundary problems for wave-structure interactions in shallow-water: DG-ALE description and local subcell correction*. JSC, 2023.
- **A. Haidar**, **F. Marche** and **F.V**, *A robust DG-ALE formulation for nonlinear shallow-water interactions with a floating object*. JSC, 2024.
- **F.V**, *Monolithic local subcell DG/FV convex property preserving scheme on unstructured grids and entropy consideration.* JCP, 2024.