Multi-scale finite elements and applications in fluid mechanics

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Context: very heterogeneous media

General idea of the Multi-scale finite elements

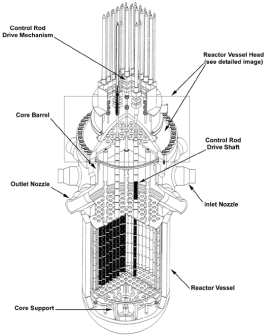
An abstract setting for the construction of such methods

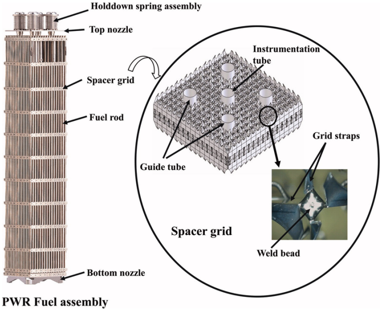
Example in 1D; interesting consequences for standard FEs; (new) Petrov-Galerkin methods for convection diffusion

Extension to multi-D with non-conforming finite elements for fluid flow problems

Conclusions and perspectives

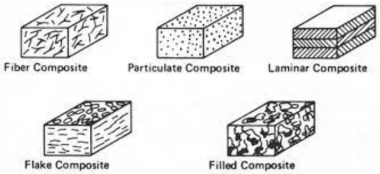
Context

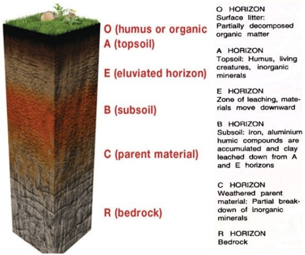




Nuclear reactor core: a multiscale medium with many obstacles.
Very expensive but still limited simulations on a partial, simplified assembly.
Is it possible to get an accurate CFD simulation (i.e. Navier-Stokes, possibly with turbulence models) of the entire core ?

Context (continued)





Composite materials

Underground

Context (continued)

Refrain from the idea that a coarse mesh will give usefull (although not very accurate) information:

On $D = (0;1)$, consider the $H_0^1(D)$ solution of

$$-\partial_x(A(x)\partial_x u) = f$$

with $A(x) = \dfrac{1}{2+\cos(2\pi x/\varepsilon)}$ and $f = 1$. The exact solution is $u_A(x) = x - x^2 + \mathcal{O}(\varepsilon)$.
But if $H = n\varepsilon$ with $n \in \mathbf{N}^*$, then a P^1 Lagrange FE will not distinguish this from the solution of the same PDE with $B(x) = \frac{1}{\sqrt{3}}$ (because A and B have the same integral on mesh segments), the solution of which is $u_B(x) = \frac{\sqrt{3}}{2}(x - x^2)$.

General idea

Consider the variational formulation: look for $u \in H_0^1(\Omega)$ s.t.

$$a(u, w) = \ell(w) \;\; \forall w \in H_0^1(\Omega) \;\; \text{with} \;\; a(u, w) = \int_\Omega A(x)\nabla u \cdot \nabla w.$$

Since $H_0^1(\Omega)$ has infinite dimension, we choose finite dimensional V_h and W_h (not necessarily included in $H_0^1(\Omega)$) and look for $u_h \in V_h$ s.t.

$$a_h(u_h, w_h) = \ell_h(w_h) \;\; \forall w_h \in W_h.$$

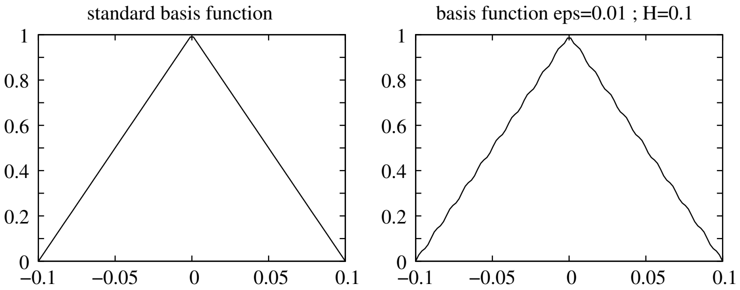
(Galerkin method if $W_h = V_h$, Petrov-Galerkin if not). And V_h approximates well $H_0^1(\Omega)$ when $h \longrightarrow 0$.

Finite elements are special cases: defined on a mesh with piecewise polynomial, compactly supported basis functions.
But you can use something else!!! The idea is that basis functions solve the original PDE on each element of the mesh, with special boundary conditions (Efendiev and Hou, Multiscale finite element methods. Theory and applications. 2009).

General idea (continued)

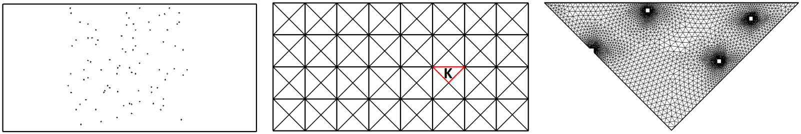
Consider $-\partial_x(A(x)\partial_x u) = f$ with $A(x) = \dfrac{1}{2+\cos(2\pi\frac{x}{\varepsilon})}$.

Basis functions solve $-\partial_x(A(x)\partial_x \phi_i) = 0$ with $\phi_i(x_j) = \delta_i^j$.



When $f \equiv 1$ and $\frac{H}{\varepsilon} \in \mathbb{N}$ using standard basis function underestimates the exact solution by a factor $\sqrt{3}/2$, while the modified method interpolates it exactly at the nodes (special case in 1D).

General idea (continued)



Heterogeneous domain — Coarse mesh — Fine mesh

The MsFEM:

- (Petrov-)Galerkin method on the coarse mesh,
- Basis/Test functions will not be standard (polynomial) FE basis functions,
- They solve a certain PDE on each element of the coarse mesh with special boundary conditions,
- Approximations of these basis functions are computed on the fine mesh (e.g. by standard FEs).

General idea (continued)

Assume a mesh with N coarse cells; each made of n fine cells. Instead of a global problem with size $\mathcal{O}(N \times n)$, the multi-scale strategy has the following advantages:

- Meshing can be done in parallel, at least for non-conforming methods as will be discussed here,
- Solutions of $\mathcal{O}(N)$ local problems of size $\mathcal{O}(n)$ may be performed in parallel, since they are independent,
- Assembling the coarse problem can be done in parallel,
- Solution of coarse problem of size $\mathcal{O}(N)$ is inexpensive,
- Localised post-processing and visualisation of results is also without extra cost.

Assume the variational formulation of the original (linear) problem reads: find $u \in V$ s.t. $a(u, w) = \ell(w)$, $\forall w \in V$.

The idea is to decompose $u = u_H + u_0$, with $u_H \in V_H$ the space of "resolved parts" and $u_0 \in V_0$ the space of "unresolved parts".

Then $a(u, w_H) = a(u_H, w_H) + a(u_0, w_H) = \ell(w_H)$, $\forall w_H \in W_H$.

And if $a(u_0, w_H) = 0$ then we can "forget" the unresolved part u_0 and solve for the resolved part u_H through:

$$\text{Find } u_H \in V_H \text{ s.t. } a(u_H, w_H) = \ell(w_H) \ , \ \forall w_H \in W_H$$

provided this problem has a (unique) solution.

So the steps of the method are

- Choose an (infinite dimensional) space V_0 of unresolved parts of the solution,
- Identify the "orthogonal" space W_H such that $a(v_0, w_H) = 0$ for all $(v_0, w_H) \in V_0 \times W_H$,
- Find an appropriate supplementary subspace V_H of V_0 such that the problem

$$\text{Find } u_H \in V_H \text{ s.t. } a(u_H, w_H) = \ell(w_H) \ , \ \forall w_H \in W_H$$

  has a unique solution,
- Identify and (numerically) construct basis of V_H and W_H and write the (Petrov-)Galerkin method in $V_H \times W_H$.

Consider $-\partial_x(\nu(x)\partial_x u) + a(x)\partial_x u + r(x)u = f$ (+ HD BC).

Multi-scale may be needed because coefficients ν, a and r may be changing on a (too) fine scale.

Let us first consider a coarse mesh $\mathcal{T}_h = \cup_i T_i = \Omega$, with $T_i = [x_{i-1}, x_i]$ and $i \in [1, N]$.

- We set $V_0 = \{v \in H_0^1(\Omega), v(x_i) = 0\,, \forall i \in [0, N]\}$. (! Works only in 1D !)

If $u = u_H + u_0$ then $u_H(x_i) = u(x_i)$. We'll have that u_H has the exact values of u at the vertices.

- For $a(u, v) = \int_\Omega (\nu u'v' + au'v + ruv)(x)dx$, what will be W_H?

(Recall that we need to have $a(u_0, w_H) = 0$, for all $u_0 \in V_0$ and all $w_h \in W_H$)

Sufficient condition (it can be proved that it is also necessary): $(w_H)_{|T_i}$ satisfying $-(\nu w_H')' - (a w_H)' + r w_H = 0$ on each T_i and continuity at the nodes. Indeed, Let $u_0 \in V_0$:

$$
\begin{aligned}
a(u_0, w_H) &= \sum_i \int_{T_i} (\nu u_0' w_H' + a u_0' w_H + r u_0 w_H)(x)dx \\
&= \sum_i \left[(\nu w_H' + a w_H)\, u_0\right]_{x_{i-1}}^{x_i} \\
&+ \sum_i \int_{T_i} (-(\nu w_H')' - (a w_H)' + r w_H) u_0(x)dx = 0.
\end{aligned}
$$

A basis $(\phi_i)_{i\in[1,N-1]}$ of W_H can be constructed by (numerically) solving $-(\nu\phi_i')' - (a\phi_i)' + r\phi_i = 0$ on $[x_{i-1}, x_i]\bigcup[x_i, x_{i+1}]$ with $\phi_i(x_j) = \delta_i^j$.

Unless $a = 0$ (vanishing convection), there is no reason to choose $V_H = W_H$. We choose

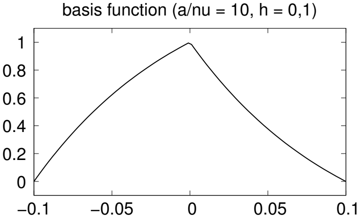
$$V_H = \{v_H \in H_0^1(\Omega) \text{ s.t. } -(\nu v_H')' + a w_H' + r w_H = 0 \text{ on } T_i\}$$

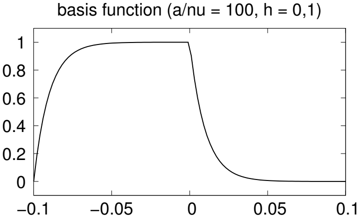
A basis $(\psi_i)_{i \in [1,N-1]}$ of V_H can be constructed by (numerically) solving $-(\nu \psi_i')' + a\psi_i' + r\psi_i = 0$ on $[x_{i-1}, x_i] \bigcup [x_i, x_{i+1}]$ with $\psi_i(x_j) = \delta_i^j$.

Solvability of the Petrov-Galerkin scheme? Yes according to Altmann, Henning, Peterseim, Acta Numerica 2021 (very ggod paper on numerical homogenization and multi-scale analysis)

Consequences for $-\nu u'' + au' = f$ + Dirichlet B.C.

Test functions of W_H solve $-\nu\phi_i'' - a\phi' = 0$ with $\phi_i(x_j) = \delta_i^j$.
Basis functions of V_H solve $-\nu\psi_i'' + a\psi' = 0$ with $\psi_i(x_j) = \delta_i^j$.





Coefficients of the associated matrix are

$$A_{i,i-1} = -\frac{a}{1-\exp(-ah/\nu)}\ ,\ A_{i,i+1} = -\frac{a\exp(-ah/\nu)}{1-\exp(-ah/\nu)}$$

$A_{i,i} = |A_{i,i-1}| + |A_{i,i+1}|$. No oscillations whatever the size of $\frac{ah}{\nu}$!
And if (f, ϕ_i) may be computed exactly, we find that
$u_H(x_i) = u(x_i)$.

Here ϕ_i solves $-\phi_i'' = 0$ and so ϕ_i is the standard P_1 Lagrange hat function (V_H is the P_1 Lagrange FE space).

As a consequence, if you are able to evaluate exactly (f, ϕ_i) then $u_H(x_i) = u(x_i)$ (which is a known fact using a Green function analysis). The error $e = (u - u_H)$ solves $-e'' = f$ on each T_i with $e(x_i) = 0$ for all i, so:

- The solution may be refined by solving these independent problems (in parallel).

- The numerical analysis is easy (don't need to use Céa's lemma nor Aubin-Nitsche's trick nor any knowledge on polynomial approximation, only Poincaré's inequality):

$||e'||_{T_i}^2 = (f, e)_{T_i} \leq ||f||_{T_i}||e||_{T_i} \leq CH||f||_{T_i}||e'||_{T_i}$.
And so $||e'||_{T_i} \leq CH||f||_{T_i}$ and $||e||_{T_i} \leq C^2H^2||f||_{T_i}$.

- Choose $p \in \mathbb{N}$. We set $V_0^{(p)} = \{v \in H_0^1(\Omega), v(x_i) = 0\,, \forall i \in [0, N]$ and $\int_{T_i} v(x)x^k dx = 0, \forall (i,k) \in [1,N] \times [0,p]\}$.

- What will be $W_H^{(p)}$?
If w_H in $H_0^1(\Omega)$ is such that $(w_H)_{|T_i}$ is a polynomial of order $p+2$, then for $u_0 \in V_0$

$$\int_\Omega u_0' w_H'(x)dx = \sum_i \left(-\int_{T_i} u_0 w_H''(x)dx + [u_0 w_H']_{x_{i-1}}^{x_i} \right) = 0.$$

Thus $W_H^{(p)}$ is the P_{p+2} Lagrange FE. If evaluation of (f, w_H) is exact, u_H will have exact values at x_i and exact moments against all polynomials of degree $\leq p$ over each T_i.

Since the operator is symmetric, we choose $V_H^{(p)} = W_H^{(p)}$.

The Stokes problem with unknowns $\mathbf{u} \in H_0^1(\Omega)^d$, $p \in L_0^2(\Omega)$.

$$c((\mathbf{u}, p), (\mathbf{v}, q)) := \nu(\nabla\mathbf{u}, \nabla\mathbf{v}) - (p, \nabla \cdot \mathbf{v}) - (\nabla \cdot \mathbf{u}, q) = (\mathbf{f}, \mathbf{v}).$$

Based on a coarse mesh with elements T and faces E, we set

$$V^{ext} = \left\{ \begin{array}{l} \mathbf{u} \in \left(L^2\left(\Omega^\varepsilon\right)\right)^d \text{ s. t. } \mathbf{u}\mid_T \in \left(H^1\left(T \cap \Omega^\varepsilon\right)\right)^d \; \forall T \\ \mathbf{u} = \mathbf{0} \text{ on } \partial\Omega^\varepsilon, \; \int_{E\cap\Omega^\varepsilon} [[\mathbf{u}]] \cdot \omega_{E,j} = 0 \, \forall E, \; j = 1, \cdots, s \end{array} \right\},$$

The space V^{ext} extends the natural velocity space $\left(H_0^1\left(\Omega^\varepsilon\right)\right)^d$; the MsFEM is nonconforming.
$\mathrm{Span}(\omega_{E,j}) = (P_n(E))^d$. The case $n = 0$ was treated by Muljadi, Narski, Lozinski, Degond (2015) (see also Le Bris, Legoll, Lozinksi, 2012, 2014 for elliptic problems). We introduced and treated the theory for the case $n > 0$, (Feng, Allaire, Omnes, 2022), and implemented $n = 1, 2$ (Balazi).

- Definition of spaces of "unresolved" fields:

$$V_0 = \left\{ \begin{array}{l} \mathbf{u} \in V_H^{ext} \text{ s. t. } \int_{E\cap\Omega^\varepsilon} \mathbf{u}\cdot\omega_{E,j} = 0, \ \int_{T\cap\Omega^\varepsilon} \mathbf{u}\cdot\varphi_{T,k} = 0, \\ \forall T \in \mathcal{T}_H, \ \forall E \in \mathcal{E}_H, \ j = 1,\cdots,s, \ k = 1,\cdots,r. \end{array} \right\},$$

$$M_0 = \left\{ p \in M \text{ s. t. } \int_{T\cap\Omega^\varepsilon} p\varpi_{T,j} = 0, \ \forall T \in \mathcal{T}_H, \ j = 1,\cdots,t \right\}.$$

The right combination is: $\text{Span}(\omega_{E,j}) = (P_n(E))^d$, $\text{Span}(\varphi_{T,k}) = (P_{n-1}(T))^d$ and $\text{Span}(\varpi_{T,j}) = P_n(T)$.
Enriching only the set of edge weights $\omega_{E,j}$ is insufficient: a given function $\mathbf{u}$ vanishing on the edges of any T would belong to the unresolved fine scales whatever the number of edge weights.

Since V_H^{ext} is not included in $(H_0^1(\Omega))^d$, we define the "broken" bilinear form c_H by

$$c_H((\mathbf{u},p),(\mathbf{v},q)) = \sum_{T\in\mathcal{T}_H} \int_{T\cap\Omega^\varepsilon} \left(\mu\nabla\mathbf{u} : \nabla\mathbf{v} - p\nabla\cdot\mathbf{v} - q\nabla\cdot\mathbf{u}\right).$$

- Space X_H that is "orthogonal" to $X_0 := V_0 \times M_0$ i.e. $(\mathbf{u},p) \in X_h \Longleftrightarrow c_H((\mathbf{u},p),(\mathbf{v},q)) = 0\ ,\ \forall(\mathbf{v},q) \in X_0$:

$$M_H = \{q \in M \text{ s. t. } q\,|_T \in \mathbb{P}_n(T),\ \forall T \in \mathcal{T}_H\}\, , = \mathbb{P}_n^{\text{disc}}$$

$$V_H = \left\{ \begin{array}{l} \mathbf{v} \in \left(L^2\left(\Omega^\varepsilon\right)\right)^d : \forall T \in \mathcal{T}_H,\ \exists \zeta^T \in M_0(T) \text{ such that} \\ -\mu\Delta\mathbf{v} + \nabla\zeta^T \in \text{span}\left\{\varphi_{T,1},\cdots,\varphi_{T,r}\right\} \text{ in } T\cap\Omega^\varepsilon \\ \nabla\cdot\mathbf{v} \in \text{span}\left\{\varpi_{T,1},\cdots,\varpi_{T,t}\right\} \text{ in } T\cap\Omega^\varepsilon \\ \mathbf{v} = \mathbf{0} \text{ on } \partial B^\varepsilon \cap T \\ \mu\nabla\mathbf{v}\mathbf{n} - \zeta^T\mathbf{n} \in \text{span}\left\{\omega_{E,1},\cdots,\omega_{E,s}\right\} \\ \text{ on } E\cap\Omega^\varepsilon\ \forall E \in \mathcal{E}(T) \end{array} \right\},$$

$$X_H = \text{span}\left\{\left(\mathbf{u}_H, \pi_H\left(\mathbf{u}_H\right) + \bar{p}_H\right), \mathbf{u}_H \in V_H,\ \bar{p}_H \in M_H\right\}.$$

with $\pi_H(\mathbf{u}_H)_{|T} := \zeta^T$ of the definition above.
Basis function $\mathbf{\Phi}_{E,i}$ associated to edge E and weight $\boldsymbol{\omega}_{E,i}$:

$$\begin{cases} -\mu\Delta\mathbf{\Phi}_{E,i} + \nabla\pi_{E,i} \in \text{span}\left\{\boldsymbol{\varphi}_{T_k,1}, \cdots, \boldsymbol{\varphi}_{T_k,r}\right\} \text{ in } T_k \cap \Omega^\varepsilon, \\ \text{div } \mathbf{\Phi}_{E,i} \in \text{span}\left\{\varpi_{T_k,1}, \cdots, \varpi_{T_k,t}\right\} \text{ in } T_k \cap \Omega^\varepsilon, \\ \mu\nabla\mathbf{\Phi}_{E,i}\boldsymbol{n} - \pi_{E,i}\boldsymbol{n} \in \text{span}\left\{\boldsymbol{\omega}_{F,1}, \cdots, \boldsymbol{\omega}_{F,s}\right\} \text{ on } F \cap \Omega^\varepsilon,\ \forall F \in \mathcal{E}(T_k), \\ \mathbf{\Phi}_{E,i} = \mathbf{0} \text{ on } \partial B^\varepsilon \cap T_k, \\ \int_{F\cap\Omega^\varepsilon} \mathbf{\Phi}_{E,i} \cdot \boldsymbol{\omega}_{F,j} = \begin{cases} \delta_{ij}, & F = E \\ 0, & F \neq E \end{cases} \quad \forall F \in \mathcal{E}\left(T_k\right),\ \forall j = 1, \cdots, s, \\ \int_{T_k\cap\Omega^\varepsilon} \mathbf{\Phi}_{E,i} \cdot \boldsymbol{\varphi}_{T_k,l} = 0\ \forall l = 1, \cdots, r, \\ \int_{T_k\cap\Omega^\varepsilon} \pi_{E,i} \cdot \varpi_{T_k,m} = 0\ \forall m = 1, \cdots, t. \end{cases}$$

Basis function $\boldsymbol{\Psi}_{T,k}$ associated to element T and weight $\boldsymbol{\varphi}_{T,k}$:

$$\begin{cases} -\mu\Delta\boldsymbol{\Psi}_{T,k} + \nabla\pi_{T,k} \in \text{span}\left\{\boldsymbol{\varphi}_{T,1},\cdots,\boldsymbol{\varphi}_{T,r}\right\} \text{ in } T\cap\Omega^\varepsilon, \\ \text{div }\boldsymbol{\Psi}_{T,k} \in \text{span}\left\{\varpi_{T,1},\cdots,\varpi_{T,t}\right\} \text{ in } T\cap\Omega^\varepsilon, \\ \mu\nabla\boldsymbol{\Psi}_{T,k}\boldsymbol{n} - \pi_{T,k}\boldsymbol{n} \in \text{span}\left\{\boldsymbol{\omega}_{F,1},\cdots,\boldsymbol{\omega}_{F,s}\right\} \text{ on } F\cap\Omega^\varepsilon,\ \forall F\in\mathcal{E}(T), \\ \boldsymbol{\Psi}_{T,k} = 0 \text{ on } \partial B^\varepsilon\cap T, \\ \int_{F\cap\Omega^\varepsilon}\boldsymbol{\Psi}_{T,k}\cdot\boldsymbol{\omega}_{F,j} = 0\ \forall F\in\mathcal{E}\left(T\right),\ \forall j=1,\cdots,s, \\ \int_{T\cap\Omega^\varepsilon}\boldsymbol{\Psi}_{T,k}\cdot\boldsymbol{\varphi}_{T,l} = \delta_{kl}\ \forall l=1,\cdots,r, \\ \int_{T\cap\Omega^\varepsilon}\pi_{T,k}\cdot\varpi_{T,m} = 0\ \forall m=1,\cdots,t. \end{cases}$$

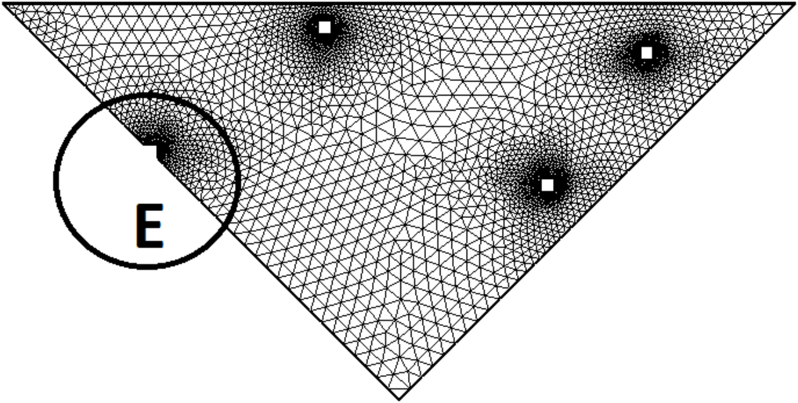
Because of the extra Lagrange multiplier, special FEs are needed to solve these problems (see L. Balazi)

- $n = 0$: Pressures $(\bar{p}_H)_T$ are in $P_0(T)$ (as in CR). The Stokes like problem reads on each T

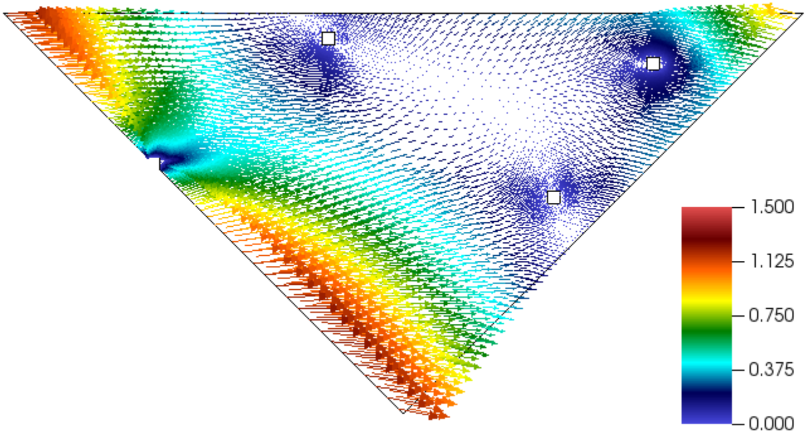
$$\begin{cases} -\mu\Delta\mathbf{v} + \nabla\zeta^T = \mathbf{0}, \\ \nabla\cdot\mathbf{v} = \alpha \in \mathbb{R}, \\ \int_T \zeta^T = 0, \\ \mathbf{v} = \mathbf{0} \text{ on } \partial\Omega^\varepsilon \cap T, \\ \mu\nabla\mathbf{v}\,\mathbf{n} - \zeta^T\mathbf{n} = \lambda \in \mathbb{R}^d, \end{cases}$$

If no obstacle, $\mathbf{v} \in P_1(T)$ and $\zeta^T = 0$. We recover the low-order Crouzeix-Raviart P_1^{NC}/P_0 scheme.

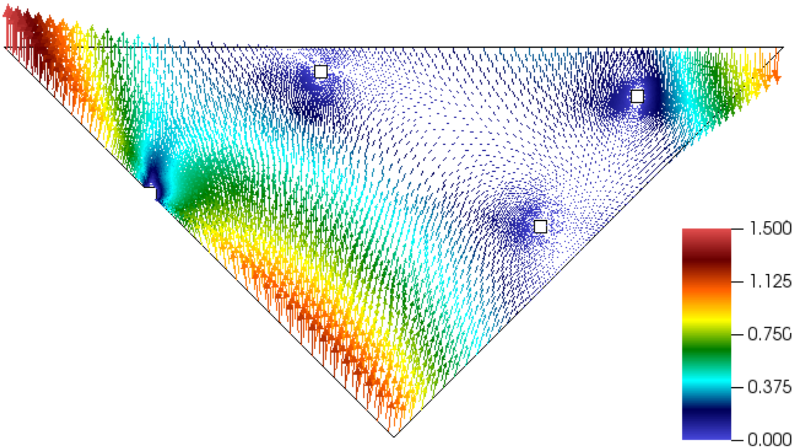
With obstacles in T:



Multi-D fluid flows (Continued)



$$\int_E \mathbf{u}_{E,1} \cdot \mathbf{e}_1 = 1, \int_E \mathbf{u}_{E,1} \cdot \mathbf{e}_2 = 0, \int_{E'} \mathbf{u}_{E,1} \cdot \mathbf{e}_j = 0.$$



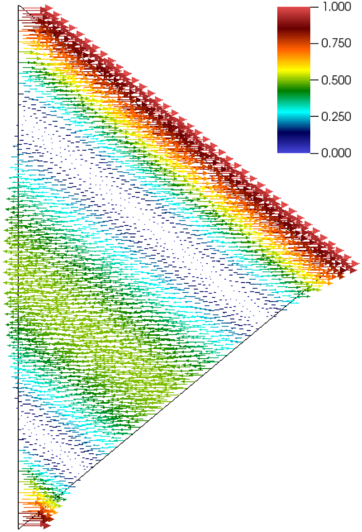
$\int_E \mathbf{u}_{E,2} \cdot \mathbf{e}_2 = 1,\ \int_E \mathbf{u}_{E,2} \cdot \mathbf{e}_1 = 0,\ \int_{E'} \mathbf{u}_{E,2} \cdot \mathbf{e}_j = 0.$

Back to the resolved spaces:

- $n = 1$: Pressures $(\bar{p}_H)_T$ are in $P_1^{\text{disc}}(T)$. The Stokes like problem reads on each T

$$\begin{cases} -\mu\Delta\mathbf{v} + \nabla\zeta^T = \mu \in \mathbb{R}^d, \\ \nabla\cdot\mathbf{v} \in P_1(T), \\ \int_T \zeta^T q = 0, \forall q \in P_1(T) \\ \mathbf{v} = \mathbf{0} \text{ on } \partial\Omega^\varepsilon \cap T, \\ \mu\nabla\mathbf{v}\,\mathbf{n} - \zeta^T\mathbf{n} \in (P_1(E))^d, \end{cases}$$

If no obstacle, $\mathbf{v} \in P_2(T)$ and $\zeta^T = 0$ belong to the space, but they are not alone.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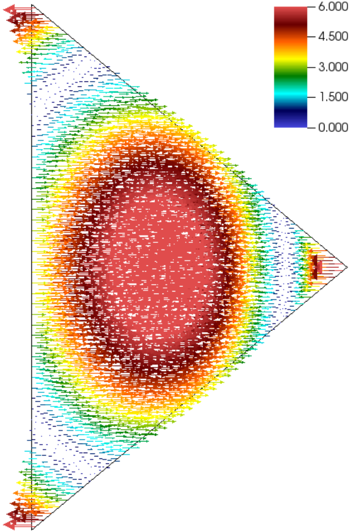
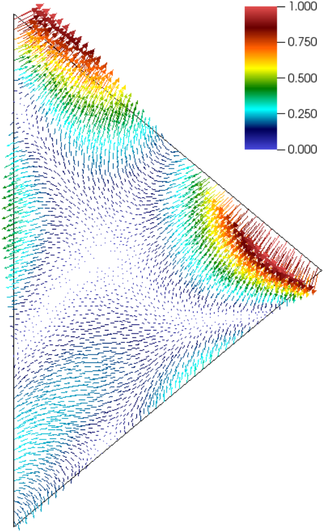


$\int_E \mathbf{u} \cdot \omega_{E,j} = 0, \ \int_T \mathbf{u} \cdot \mathbf{e}_1 = 1, \ \int_T \mathbf{u} \cdot \mathbf{e}_2 = 0.$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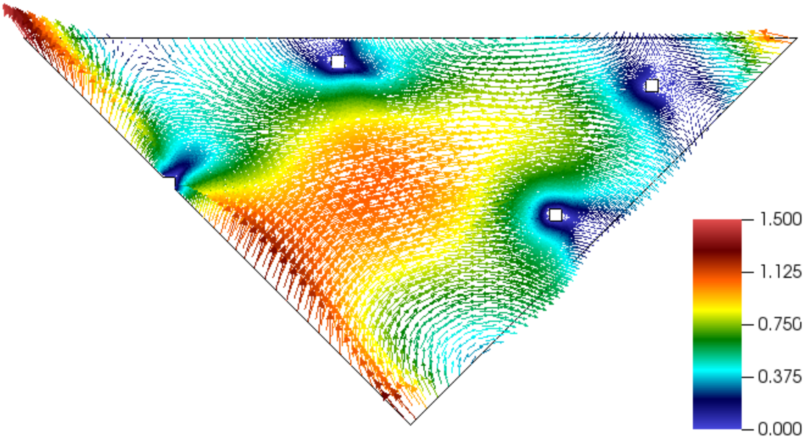
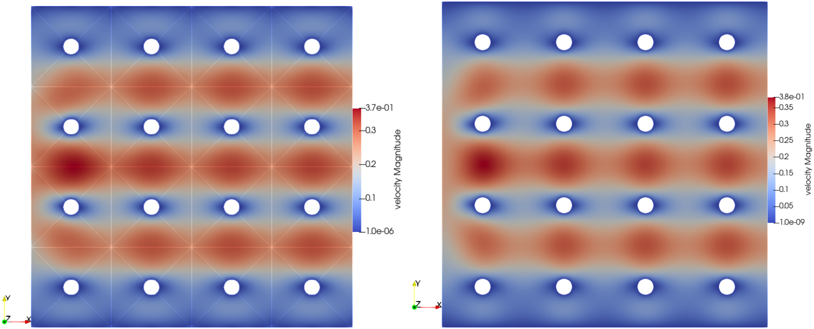
Multi-D fluid flows (Continued)

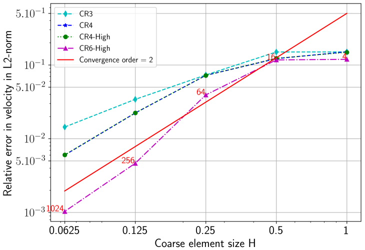


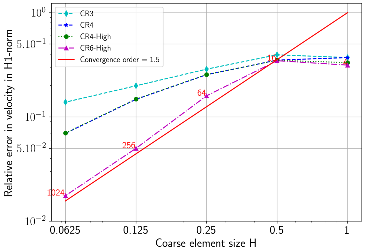
Multi-D fluid flows (Continued)

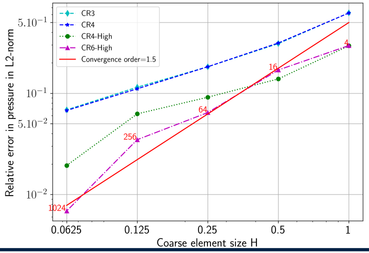


Channel flow with obstacles and $4 \times 4 \times 4$ coarse triangles $n = 2$ and reference calculation

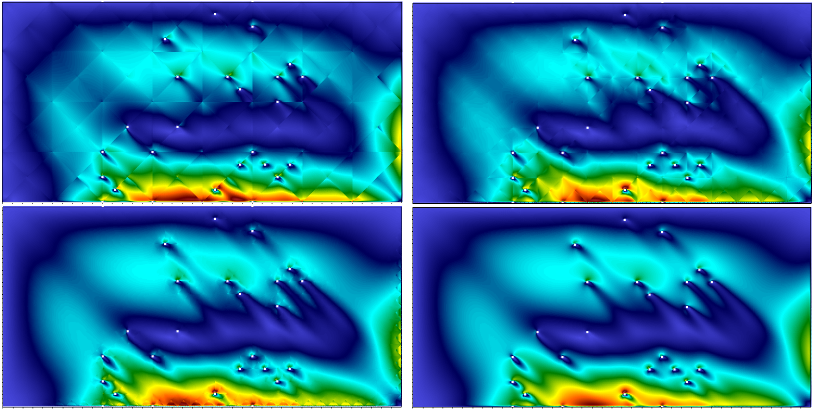
Multi-D fluid flows (Continued)







Oseen flow with $U_0 = (400, -400)$ and 26 obstacles of size 1.5×10^{-2}



$n = 1$, from left to right, top to bottom: $4 \times 8 \times 4$ coarse triangles, $8 \times 16 \times 4$ triangles, $16 \times 32 \times 4$ triangles, reference on a fine mesh with 2 million triangles.

Conclusions and perspectives

- Construction of Multi-Scale Finite Elements
- Provides new insight on Standard Finite Elements
- Can reproduce complex fluid flows with low cost

- More tests with Oseen and Navier-Stokes flows
- A priori and a posteriori analysis including effects of approximate resolution of fine-scale basis functions
- Non-stationary problems
- Non-linear problems: turbulence modelling
- Links with Domain Decomposition