Global entropy stability for a class of unlimited high-order schemes for hyperbolic systems of conservation laws

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Context

Hyperbolic systems of conservation laws

For the unknown $w(x,t) \in \Omega \subset \mathbb{R}^d$, we consider the Cauchy problem

$$\partial_t w + \partial_x f(w) = 0, \qquad (x,t) \in \mathbb{R} \times (0,+\infty)$$
$$w(x, t = 0) = w_0(x)$$

With the Jacobian $\nabla f(u)$ diagonalizable in $\mathbb{R}$ so that the system is hyperbolic

- For genuine non linear equations, the solutions w may develop discontinuities
- Dicontinuities may implay lost of uniqueness
- To recover uniqueness $\longrightarrow$ Introduction of local entropy inequalities

  $$\partial_t \eta(w) + \partial_x Q(w) \leqslant 0 \text{ in } \mathcal{D}'((0,+\infty) \times \mathbb{R})$$

  for any pair entropy-entropy flux $\eta \in C^1(\Omega;\mathbb{R})$ convex and $Q \in C^1(\Omega;\mathbb{R})$ defined by

  $$\nabla \eta(w)^T \nabla f(w) = \nabla Q(w)^T$$

- Weak entropy principle (global)

  $$\int_{\mathbb{R}} \eta(w(x,t))dx \leqslant \int_{\mathbb{R}} \eta(w(x,s))dx \qquad t > s \geqslant 0$$

Finite-volume schemes and space-consistency

Mesh definition: $\Delta x, \Delta t > 0$, $C_i := (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$, $x_{i\pm\frac{1}{2}} = i\Delta x \pm \frac{\Delta x}{2}$, $t^n = n\Delta t$

Numerical approximation

$$\forall n \in \mathbb{Z}, i \in \mathbb{Z}, \quad \frac{w_i^{n+1} - w_i^n}{\Delta t} + \frac{F^n_{i+\frac{1}{2}} - F^n_{i-\frac{1}{2}}}{\Delta x} = 0$$

where $w_i^n \approx \frac{1}{\Delta x}\int_{C_i} w(t^n, x)dx$.

Definition (Formal consistency)

Let $r \in \mathbb{N}^*$ and $F : \Omega^{2r+2} \to \mathbb{R}$ a continuous function such that $F(w, ..., w) = f(w)$. A finite volume scheme with a numerical flux $F^n_{i+\frac{1}{2}} = F(w^n_{i-r}, ..., w^n_{i+r+1}))$ is formally second-order consistent in space if for a smooth compactly supported function $w(x)$ and for all $i \in \mathbb{Z}$,

$$F_{i+\frac{1}{2}} = f(w(x_{i+\frac{1}{2}})) + \mathcal{O}(\Delta x^2), \quad w_i = \frac{1}{\Delta x}\int_{C_i} w(x)dx,$$

where the function $\mathcal{O}(\Delta x^2)$ is independent on $i \in \mathbb{Z}$.

- The centered flux $F^n_{i+\frac{1}{2}} = \frac{f(w_i^n)+f(w^n_{i+1})}{2}$ is second-order (but not stable)
- The HLL flux $F^n_{i+\frac{1}{2}} = \frac{f(w_i^n)+f(w^n_{i+1})}{2} - \frac{\lambda}{2}(w^n_{i+1} - w_i^n)$ is first-order.

The litterature

Challenge: design a **true high-order** in space numerical methods that verifies a discrete entropy inequality (local or global)

- An entropy preserving 3-point scheme is at most first-order accurate
- MUSCL strategy: lost of order
- Entropy variable strategy (Tadmor): semi-discrete entropy inequality
- Nonlinear projection strategy (LefLoch, Coquel)
- Global entropy estimations (Mishra, Parés)
- Artificial viscosity (Abgrall, Tadmor)
- Implicit schemes (Dumbser)
- Multi-cells entropy inequalities (Ricchiuto)

Second-order in space dissipative schemes

A second-order in space finite-volume scheme

High-order HLL scheme

$$F^{O2}_{i+\frac{1}{2}} := \frac{f(w_i^n) + f(w_{i+1}^n)}{2} - \frac{\lambda}{2}(w_{i+1}^n - w_i^n) + \frac{(\alpha_i^{O2} + \alpha_{i+1}^{O2})}{2}$$

$(\alpha_i)_{i\in\mathbb{Z}}$ ensures the formal second-order consistency of the flux
For a smooth function $w(x)$, set

$$w_i := \frac{1}{\Delta x}\int_{C_i} w(x)dx$$

Taylor expansion around $x_{i+\frac{1}{2}}$ yields

$$F^{O2}_{i+\frac{1}{2}} = f(w(x_{i+\frac{1}{2}})) - \frac{\lambda\Delta x}{2}\partial_x w(x_{i+\frac{1}{2}}) + \mathcal{O}(\Delta x^2) + \frac{(\alpha_i^{O2} + \alpha_{i+1}^{O2})}{2}.$$

The formal second-order consistency in space is ensured provided:

$$\alpha_i^{O2} = \frac{\lambda\Delta x}{2}\partial_x w(x_{i+\frac{1}{2}}) + \mathcal{O}(\Delta x^2).$$

The approximation formulas of the spatial derivative of $\partial_x w(x_{i+\frac{1}{2}})$ are degrees of freedom.

Definition of the corrective terms

Key idea: Choose $(\alpha_i)_{i\in\mathbb{Z}}$ in such a way that:

- The formal second order consistency statement holds:

$$\alpha_i^{O2} = \frac{\lambda\Delta x}{2}\partial_x w(x_{i+\frac{1}{2}}) + \mathcal{O}(\Delta x^2).$$

- The following dissipativity inequality relatively to a given entropy η is verified for non trivial state $(w_i^n)_{i\in\mathbb{Z}}$:

$$\sum_{i\in\mathbb{Z}} \left(F^{O2}_{i+\frac{1}{2}} - F^{O2}_{i-\frac{1}{2}} \right) \cdot \nabla\eta(w_i^n) > 0.$$

A possibility (among many others) is to set:

$$\alpha_i^{O2} := \frac{\lambda\Delta x \partial_x w_i^{O2}}{2}$$

$$\Delta x\partial_x w_i^{O2} := \Theta_i\delta_{i+\frac{1}{2}} + (1-\Theta_i)\delta_{i-\frac{1}{2}}, \quad \delta_{i+\frac{1}{2}} = w^n_{i+1} - w^n_i$$

where $(\Theta_i)_{i\in\mathbb{Z}}$ is a sequence of diagonal matrices bounded as $\Delta x \to 0$ chosen such that the entropy dissipation inequality holds

Main result

Theorem

Consider $\eta \in C^2(\Omega, \mathbb{R})$ a strictly convex entropy. Let the approximation at time t^n, $(w_i^n)_{i\in\mathbb{Z}}$ being a non zero sequence in $h^2(\mathbb{Z})$ and such that $\sum_{i\in\mathbb{Z}} \eta(w_i^n)\Delta x$ is finite. We assume the following:

a) *There exists a compact set $K \subset \Omega$ such that $w_i^n \in K$ for every $i \in \mathbb{Z}$.*

b) *The sequence of bounded (as $\Delta x \to 0$) diagonal matrices $(\Theta_i)_{i\in\mathbb{Z}}$ satisfies for all $i \in \mathbb{Z}$ the entropy dissipative inequality*

$$\sum_{i\in\mathbb{Z}} \left(F^{O2}_{i+\frac{1}{2}} - F^{O2}_{i-\frac{1}{2}} \right) \cdot \nabla\eta(w_i^n) < 0$$

Then there exists two constants $\lambda^n > 0$ and $C^n > 0$ with $C^n \underset{\Delta x \to 0}{=} \mathcal{O}(1)$ such that for any $\lambda > \lambda^n$ and $0 < \frac{\Delta t}{\Delta x^2} \leqslant C^n$ the second-order scheme

$$\frac{w_i^{n+1} - w_i^n}{\Delta t} + \frac{F^{O2}_{i+\frac{1}{2}} - F^{O2}_{i-\frac{1}{2}}}{\Delta x} = 0.$$

satifies

$$w_i^{n+1} \in \Omega \qquad \text{and} \qquad \sum_{i\in\mathbb{Z}} \eta(w_i^{n+1})\Delta x \leqslant \sum_{i\in\mathbb{Z}} \eta(w_i^n)\Delta x$$

Main result

Comments:

- The assumption a) is technical. It can be completely removed in the case where $\Omega = \mathbb{R}^d$
- The assumption b) is the dissipative inequality relatively to an entropy η
  Lot of choices of matrices $(\Theta_i)_{i\in\mathbb{Z}}$ can be exhibited. Form scalar equation with $\eta(w) = w^2/2$

$$\Theta_i^m = -\theta_a \,\mathrm{sign}\left((\delta^m_{i+\frac{1}{2}})^2 - (\delta^m_{i-\frac{1}{2}})^2\right),$$

$$\Theta_i^m = \frac{\left((\delta^m_{i-\frac{1}{2}})^2 - (\delta^m_{i+\frac{1}{2}})^2\right)\left((\delta^m_{i-\frac{1}{2}})^2 + (\delta^m_{i+\frac{1}{2}})^2\right)}{\left((\delta^m_{i-\frac{1}{2}})^2 + (\delta^m_{i+\frac{1}{2}})^2\right)^2 + \varepsilon}$$

- The strict convexity of the entropy is needed in the proof because one needs to get a positive lower bound for the Hessian $\nabla^2\eta$.
- One can recover a hyperbolic CFL condition using a second-order in time discretization.

Main result

- Global entropy and first-order viscosity (Burgers equation and quadratic entropy)

$$\begin{aligned} \partial_t w + \partial_x w^2/2 &= \varepsilon \partial_x^2 w \\ \partial_t w^2/2 + \partial_x w^3/3 &= \varepsilon w \partial_x^2 w \\ &= \varepsilon \partial_x (w \partial_x w) - \varepsilon (\partial_x w)^2 \end{aligned}$$

so that we get

$$\frac{d}{dt} \int_{\mathbb{R}} \frac{1}{2} w^2(x,t) dx \leqslant 0$$

- Global entropy and second-order viscosity (Burgers equation and quadratic entropy)

$$\begin{aligned} \partial_t w + \partial_x w^2/2 &= -\varepsilon \partial_x^4 w \\ \partial_t w^2/2 + \partial_x w^3/3 &= -\varepsilon w \partial_x^4 w \\ &= -\varepsilon \partial_x (w \partial_x^3 w) + \varepsilon \partial_x (\partial_x w \partial_x^2 w) - \varepsilon (\partial_x^2 w)^2 \end{aligned}$$

so that we get

$$\frac{d}{dt} \int_{\mathbb{R}} \frac{1}{2} w^2(x,t) dx \leqslant 0$$

A simple proof of global entropy inequality in the case of the linear transport equation

The scalar linear transport equation

Consider the scalar linear transport equation of velocity $a \neq 0$

$$\partial_t w + a\partial_x w = 0$$
$$w(x,0) = w_0(x)$$

We consider the quadratic entropy $\eta(w) = \frac{w^2}{2}$, our result then simply states the quadratic stability. The scheme can be written in the form:

$$\frac{w_i^{n+1} - w_i^n}{\Delta t} = -\frac{a}{2\Delta x}(w_{i+1}^n - w_{i-1}^n) + \frac{\lambda}{4\Delta x}\Delta_i^n - \frac{\lambda}{4\Delta x}(\Theta_{i+1}^n \Delta_{i+1}^n + (1 - \Theta_{i-1})\Delta_{i-1}^n)$$

where $\lambda > 0$ and $\Delta_i^n = w_{i+1}^n - 2w_i^n + w_{i-1}^n$

The scalar linear transport equation

Multiply the scheme by w_i^n and sum over $i \in \mathbb{Z}$. It yields

$$\underbrace{\sum_{i\in\mathbb{Z}} \frac{w_i^{n+1} - w_i^n}{\Delta t} \cdot w_i^n}_{D_0} = \underbrace{\sum_{i\in\mathbb{Z}} -\frac{a}{2\Delta x}(w_{i+1}^n - w_{i-1}^n) \cdot w_i^n}_{D_1}$$

$$+ \underbrace{\sum_{i\in\mathbb{Z}} \frac{\lambda}{4\Delta x} \Delta_i^n \cdot w_i^n}_{D_2}$$

$$+ \underbrace{\sum_{i\in\mathbb{Z}} -\frac{\lambda}{4\Delta x}(\Theta_{i+1}\Delta_{i+1}^n + (1-\Theta_{i-1})\Delta_{i-1}^n) \cdot w_i^n}_{D_3}.$$

Using translations of indices, one rearranges the terms D_1, D_2 and D_3.

The scalar linear transport equation

We get

$$D_1 = 0,$$

$$D_2 + D_3 = -\sum_{i\in\mathbb{Z}} \frac{\lambda}{8\Delta x}|\Delta_i^n|^2 + \sum_{i\in\mathbb{Z}} \frac{\lambda}{4\Delta x}\Theta_i\Delta_i^n(w_{i+1}^n - w_{i-1}^n).$$

Comments:

- Choose Θ_i (bounded as $\Delta x \to 0$) and such that $\sum_{i\in\mathbb{Z}} \frac{\lambda}{4\Delta x}\Theta_i\Delta_i^n(w_{i+1}^n - w_{i-1}^n) \leqslant 0$. For example $\Theta_i = -\mathrm{sgn}(\Delta_i^n(w_{i+1}^n - w_{i-1}^n))$.
- A consistency analysis shows that for a smooth function $w(x)$

$$|\Delta_i^n|^2 \approx \Delta x^4 \partial_{xx} w(x_i)^2$$

$$\Delta_i^n(w_{i+1}^n - w_{i-1}^n) \approx \Delta x^3 \partial_{xx} w(x_i)\partial_x w(x_i)$$

The scalar linear transport equation

We reformulate D_0

$$\Delta t D_0 = \frac{1}{2}\sum_{i\in\mathbb{Z}}(|w_i^{n+1}|^2 - |w_i^n|^2) - \frac{1}{2}\sum_{i\in\mathbb{Z}}|w_i^{n+1} - w_i^n|^2.$$

to write

$$\begin{aligned}\frac{1}{2}\sum_{i\in\mathbb{Z}}|w_i^{n+1}|^2\Delta x - \frac{1}{2}\sum_{i\in\mathbb{Z}}|w_i^n|^2\Delta x \\ &= \frac{\Delta x}{2\Delta t}\sum_{i\in\mathbb{Z}}|w_i^{n+1} - w_i^n|^2 - \sum_{i\in\mathbb{Z}}\frac{\lambda}{8}|\Delta_i|^2 + \sum_{i\in\mathbb{Z}}\frac{\lambda}{4}\Theta_i\Delta_i^n(w_{i+1}^n - w_{i-1}^n) \\ &= \frac{\Delta t}{2\Delta x}\sum_{i\in\mathbb{Z}}|F_{i+\frac{1}{2}}^{O2} - F_{i-\frac{1}{2}}^{O2}|^2 - \sum_{i\in\mathbb{Z}}\frac{\lambda}{8}|\Delta_i^n|^2 + \sum_{i\in\mathbb{Z}}\frac{\lambda}{4}\Theta_i\Delta_i^n(w_{i+1}^n - w_{i-1}^n)\end{aligned}$$

Make appropriate regularity and summability assumption, typically $(w_i)_{i\in\mathbb{Z}} \in h^2(\mathbb{Z})$ and non zero. Consider

$$0 < \frac{\Delta t}{\Delta x} \leqslant \frac{\sum_{i\in\mathbb{Z}}\frac{\lambda}{8}|\Delta_i^n|^2 - \sum_{i\in\mathbb{Z}}\frac{\lambda}{4}\Theta_i\Delta_i^n(w_{i+1}^n - w_{i-1}^n)}{\sum_{i\in\mathbb{Z}}|F_{i+\frac{1}{2}}^{O2} - F_{i-\frac{1}{2}}^{O2}|^2}$$

so that the right hand side is positive, then one gets the quadratic stability.

The scalar linear transport equation

Comments:

- The previous consistency analysis shows that the CFL is of parabolic type

$$\frac{\sum_{i\in\mathbb{Z}} \frac{\lambda}{8}|\Delta_i^n|^2 - \sum_{i\in\mathbb{Z}} \frac{\lambda}{4}\Theta_i\Delta_i^n(w_{i+1}^n - w_{i-1}^n)}{\sum_{i\in\mathbb{Z}} |F_{i+\frac{1}{2}}^{O2} - F_{i-\frac{1}{2}}^{O2}|^2} = \mathcal{O}(\Delta x)$$

- The CFL is global and probably far from being optimal
- The proof generalizes to any strictly convex entropy η but the proof is more technical because the sum $\sum_{i\in\mathbb{Z}} a(w_{i+1} - w_{i-1}) \cdot \eta'(w_i)$ is not exactly zero (it is zero up to high order terms)

Numerical results

Burgers equation

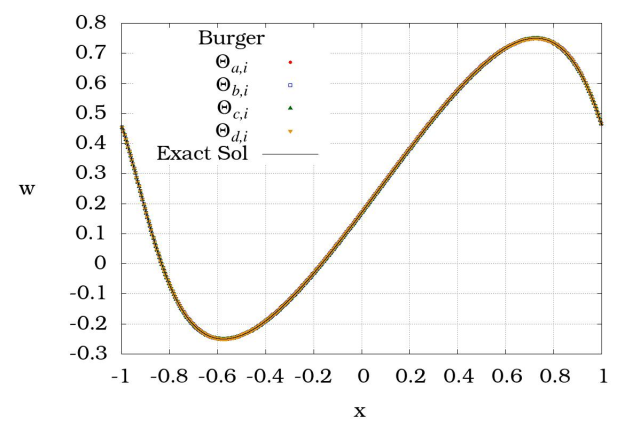
In the following experiment, we used a second-order in time discretization and fix the hyperbolic CFL condition $\frac{\lambda \Delta t}{\Delta x} \leqslant \frac{1}{2}$.

$\Omega = \mathbb{R}, f(w) = w^2/2, \eta(w) = w^2/2$. We test four choices for the sequence $(\Theta_i)_{i\in\mathbb{Z}}$.

$$
\begin{aligned}
\Theta^m_{a,i} &= -\theta_a \operatorname{sign}\left((\delta^m_{i+\frac{1}{2}})^2 - (\delta^m_{i-\frac{1}{2}})^2\right), \\
\Theta^m_{b,i} &= -\theta_b \tanh\left((\delta^m_{i+\frac{1}{2}})^2 - (\delta^m_{i-\frac{1}{2}})^2\right), \\
\Theta^m_{c,i} &= \frac{\left((\delta^m_{i-\frac{1}{2}})^2 - (\delta^m_{i+\frac{1}{2}})^2\right)\left((\delta^m_{i-\frac{1}{2}})^2 + (\delta^m_{i+\frac{1}{2}})^2\right)}{\left((\delta^m_{i-\frac{1}{2}})^2 + (\delta^m_{i+\frac{1}{2}})^2\right)^2 + \varepsilon}, \\
\Theta^m_{d,i} &= \frac{1}{2},
\end{aligned}
\tag{1}
$$

Burgers equation: smooth solution

We take a smooth initial data $w_0(x) = 0.25 + 0.5\sin(\pi x)$ over a periodic domain $[-1, 1)$. With a final time small enough, here given by $t = 0.3$, the exact solution remains smooth so that the order of accuracy can be evaluated. For a smooth periodic smooth solution, we measure the error in L^1, L^2, L^∞. We obtain second order of accuracy. Plots corresponds to 400 cells.



Burgers equation: smooth solution

	Second-order scheme errors $\Theta_i^m = \Theta_{a,i}^m$					
cells	L^1	order	L^2	order	L^∞	order
100	5.7E-04	-	6.8E-04	-	1.8E-03	-
200	1.4E-04	2.0	1.7E-04	2.0	4.5E-04	2.0
400	3.5E-05	2.0	4.1E-05	2.0	1.1E-04	2.0
800	8.8E-06	2.0	1.0E-05	2.0	2.8E-05	2.0
1600	2.2E-06	2.0	2.6E-06	2.0	6.9E-06	2.0

	Second-order scheme errors $\Theta_i^m = \Theta_{b,i}^m$					
cells	L^1	order	L^2	order	L^∞	order
100	5.7E-04	-	6.8E-04	-	1.8E-03	-
200	1.4E-04	2.0	1.7E-04	2.0	4.5E-04	2.0
400	3.5E-05	2.0	4.1E-05	2.0	1.1E-04	2.0
800	8.8E-06	2.0	1.0E-05	2.0	2.8E-05	2.0
1600	2.2E-06	2.0	2.6E-06	2.0	6.9E-06	2.0

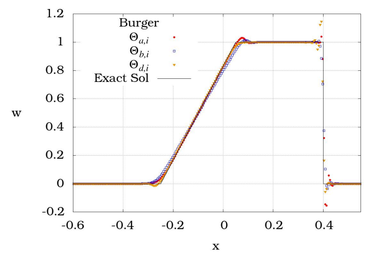
	Second-order scheme errors $\Theta_i^m = \Theta_{c,i}^m$					
cells	L^1	order	L^2	order	L^∞	order
100	1.4E-03	-	1.6E-03	-	4.1E-03	-
200	2.4E-04	2.5	2.8E-04	2.6	7.6E-04	2.4
400	3.9E-05	2.7	4.3E-05	2.7	1.1E-04	2.8
800	8.7E-06	2.1	1.0E-05	2.1	2.7E-05	2.0
1600	2.2E-06	2.0	2.6E-06	2.0	6.8E-06	2.0

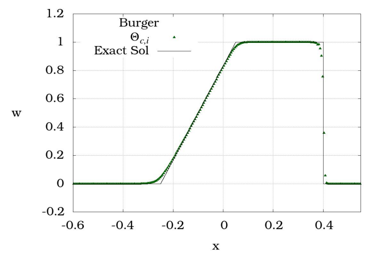
	Second-order scheme errors $\Theta_i^m = \Theta_{d,i}^m$					
cells	L^1	order	L^2	order	L^∞	order
100	4.4E-04	-	4.8E-04	-	1.1E-03	-
200	1.1E-04	2.0	1.1E-04	2.1	2.7E-04	2.1
400	2.6E-05	2.0	2.8E-05	2.0	6.6E-05	2.0
800	6.5E-06	2.0	6.8E-06	2.0	1.6E-05	2.0
1600	1.6E-06	2.0	1.7E-06	2.0	4.0E-06	2.0

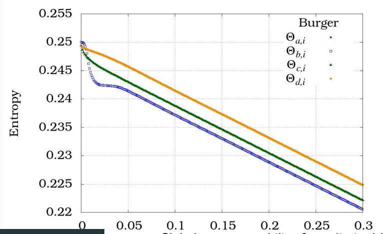
Burgers equation: discontinuous solution

We take a discontinuous initial data over the periodic domain $[-1, 1)$ defined by

$$w_0(x) = \begin{cases} 1 & \text{if} \quad -0.25 \leqslant x \leqslant 0.25, \\ 0 & \text{otherwise.} \end{cases}$$







Burgers equation: discontinuous solution

	Second-order scheme errors $\Theta_i^m = \Theta_{a,i}^m$					
cells	L^1	order	L^2	order	L^∞	order
100	3.4E-02	-	6.4E-02	-	3.3E-01	-
200	1.7E-02	1.0	4.3E-02	0.6	3.3E-01	0.0
400	8.5E-03	1.0	3.0E-02	0.5	3.2E-01	0.0
800	4.3E-03	1.0	2.1E-02	0.5	3.2E-01	0.0
1600	2.1E-03	1.0	1.5E-02	0.5	3.2E-01	0.0

	Second-order scheme errors $\Theta_i^m = \Theta_{b,i}^m$					
cells	L^1	order	L^2	order	L^∞	order
100	5.4E-02	-	9.0E-02	-	3.8E-01	-
200	2.7E-02	1.0	5.9E-02	0.6	3.8E-01	0.0
400	1.4E-02	1.0	4.0E-02	0.6	3.8E-01	0.0
800	7.0E-03	1.0	2.7E-02	0.6	3.8E-01	0.0
1600	3.5E-03	1.0	1.8E-02	0.5	3.8E-01	0.0

	Second-order scheme errors $\Theta_i^m = \Theta_{c,\varepsilon,i}^m$					
cells	L^1	order	L^2	order	L^∞	order
100	3.5E-02	-	7.1E-02	-	3.6E-01	-
200	1.8E-02	0.9	4.9E-02	0.5	3.6E-01	0.0
400	9.2E-03	1.0	3.4E-02	0.5	3.6E-01	0.0
800	4.6E-03	1.0	2.3E-02	0.5	3.6E-01	0.0
1600	2.3E-03	1.0	1.6E-02	0.5	3.6E-01	0.0

	Second-order scheme errors $\Theta_i^m = \Theta_{d,i}^m$					
cells	L^1	order	L^2	order	L^∞	order
100	3.1E-02	-	5.8E-02	-	2.8E-01	-
200	1.4E-02	1.1	4.0E-02	0.6	2.8E-01	0.0
400	7.1E-03	1.0	2.8E-02	0.5	2.8E-01	0.0
800	3.5E-03	1.0	1.9E-02	0.5	2.8E-01	0.0
1600	1.7E-03	1.0	1.4E-02	0.5	2.8E-01	0.0

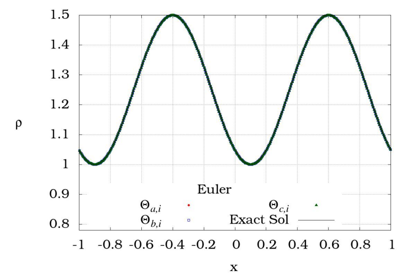
Euler equations

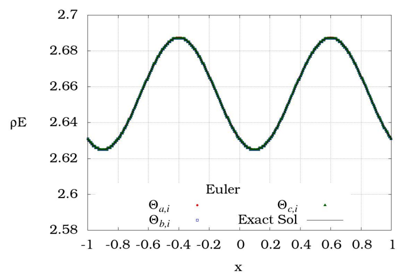
We consider the Euler system where $\Omega = \{(\rho, \rho u, \rho E) \in \mathbb{R}^3 \ : \ \rho > 0, \rho E - \rho u^2/2 > 0\}$ where the unknown vector is $w = (\rho, \rho u, \rho E)^T$ and the flux function is $f(w) = (\rho u, \rho u^2 + p, \rho E u + p u)^T$, with $p = (\gamma - 1)(\rho E - \frac{\rho u^2}{2})$. We fix $\gamma = 1.4$ and we consider the entropy $\eta(w) = -\rho \ln\left(\frac{p}{\rho^\gamma}\right)$. For the Euler problem, we use the following matrix parameter

$$
\begin{aligned}
\Theta^n_{a,i} &= -\theta_a \,\mathrm{diag}_{1\leqslant j\leqslant 3}\left(\mathrm{sign}\left(\left(\nabla\eta(w^n_{i+1}) - \nabla\eta(w^n_{i-1})\right)_j (\delta^m_{i+\frac{1}{2}} - \delta^m_{i-\frac{1}{2}})_j\right)\right),\\
\Theta^n_{b,i} &= -\theta_b \,\mathrm{diag}_{1\leqslant j\leqslant 3}\left(\tanh\left(\left(\nabla\eta(w^n_{i+1}) - \nabla\eta(w^n_{i-1})\right)_j (\delta^m_{i+\frac{1}{2}} - \delta^m_{i-\frac{1}{2}})_j\right)\right),\\
\Theta^m_{c,\varepsilon,i} &= \mathrm{diag}_{1\leqslant j\leqslant 3}\left(\frac{\left(\left(\delta^m_{i-\frac{1}{2}}\right)_j^2 - \left(\delta^m_{i+\frac{1}{2}}\right)_j^2\right)\left(\left(\delta^m_{i-\frac{1}{2}}\right)_j^2 + \left(\delta^m_{i+\frac{1}{2}}\right)_j^2\right)}{\left(\left(\delta^m_{i-\frac{1}{2}}\right)_j^2 + \left(\delta^m_{i+\frac{1}{2}}\right)_j^2\right)^2 + \varepsilon}\right),
\end{aligned}
\tag{2}
$$

Euler equation: smooth solution

We take a smooth initial data $\rho_0(x) = 1 + 0.5\sin^2(\pi x)$, $u_0(x) = 0.5$, $p_0(x) = 1$ over a periodic domain $[-1, 1)$. Fro all $t > 0$ the exact solution remains smooth so that the order of accuracy can be evaluated. We measure the error in L^1, L^2, L^∞. We obtain second order of accuracy. Plots corresponds to 400 cells.





Euler equation: smooth solution

	Second-order scheme errors $\Theta_i^m = \Theta_{a,i}^m$					
cells	L^1	order	L^2	order	L^∞	order
100	3.5E-03	-	1.9E-03	-	1.7E-03	-
200	8.7E-04	2.0	4.7E-04	2.0	4.2E-04	2.0
400	2.2E-04	2.0	1.2E-04	2.0	1.0E-04	2.0
800	5.4E-05	2.0	2.9E-05	2.0	2.6E-05	2.0
1600	1.4E-05	2.0	7.4E-06	2.0	6.5E-06	2.0

	Second-order scheme errors $\Theta_i^m = \Theta_{b,i}^m$					
cells	L^1	order	L^2	order	L^∞	order
100	3.5E-03	-	1.9E-03	-	1.7E-03	-
200	8.7E-04	2.0	4.7E-04	2.0	4.2E-04	2.0
400	2.2E-04	2.0	1.2E-04	2.0	1.0E-04	2.0
800	5.4E-05	2.0	2.9E-05	2.0	2.6E-05	2.0
1600	1.4E-05	2.0	7.4E-06	2.0	6.5E-06	2.0

	Second-order scheme errors $\Theta_i^m = \Theta_{c,\varepsilon,i}^m$					
cells	L^1	order	L^2	order	L^∞	order
100	1.2E-02	-	7.6E-03	-	1.1E-02	-
200	2.4E-03	2.3	1.6E-03	2.2	2.9E-03	1.9
400	3.4E-04	2.8	2.0E-04	3.0	3.7E-04	3.0
800	6.0E-05	2.5	3.1E-05	2.7	2.6E-05	3.8
1600	1.4E-05	2.1	7.4E-06	2.1	6.5E-06	2.0

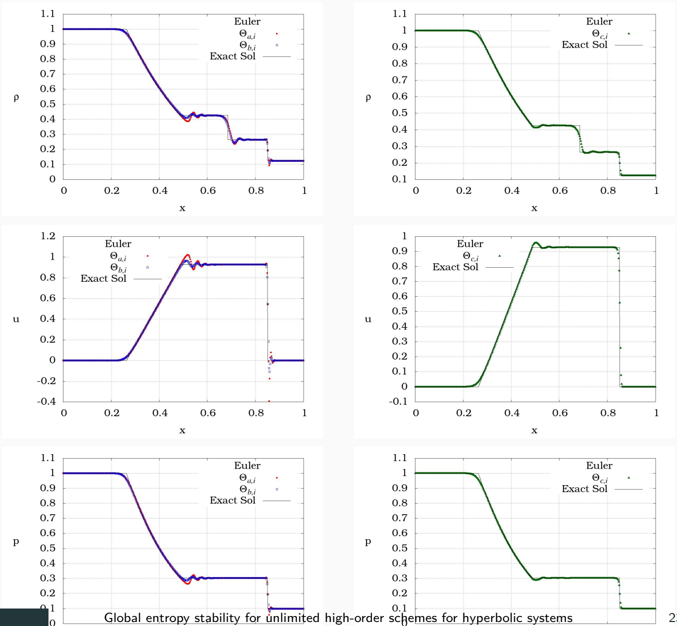
Euler equations: Shock tube solution

We consider the initial data given by

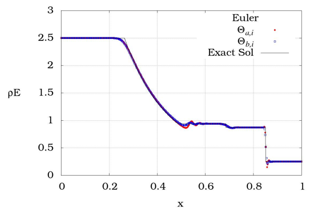
$$\rho_0(x) = \begin{cases} 1 & \text{if } x < 0.5, \\ 0.125 & \text{otherwise,} \end{cases} \qquad u_0(x) = 0, \qquad p_0(x) = \begin{cases} 1 & \text{if } x < 0.5, \\ 0.1 & \text{otherwise,} \end{cases}$$

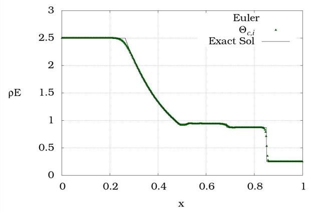
over the domain $[0, 1]$. The final time is 0.2. To respect the periodic conditions on the boundaries, we work on the domain $[-1, 1]$ and we symmetrize the shock tube problem on $[-1, 0]$.

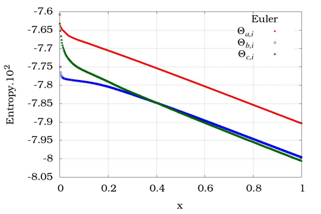
Euler equations: Shock tube solution



Euler equations: Shock tube solution







Euler equations: Shock tube solution

	Second-order scheme errors $\Theta_i^m = \Theta_{a,i}^m$					
cells	L^1	order	L^2	order	L^∞	order
100	7.2E-02	-	7.1E-02	-	2.7E-01	-
200	4.0E-02	0.8	4.5E-02	0.7	2.4E-01	0.1
400	2.2E-02	0.9	2.9E-02	0.6	2.4E-01	0.0
800	1.2E-02	0.9	1.9E-02	0.6	1.9E-01	0.4
1600	6.4E-03	0.9	1.3E-02	0.6	1.7E-01	0.1
	Second-order scheme errors $\Theta_i^m = \Theta_{b,i}^m$					
cells	L^1	order	L^2	order	L^∞	order
100	5.9E-02	-	6.2E-02	-	1.9E-01	-
200	3.3E-02	0.8	3.9E-02	0.7	1.9E-01	0.0
400	1.8E-02	0.9	2.5E-02	0.6	1.9E-01	0.0
800	9.4E-03	0.9	1.5E-02	0.7	1.3E-01	0.6
1600	5.2E-03	0.9	1.1E-02	0.5	1.6E-01	0.4
	Second-order scheme errors $\Theta_i^m = \Theta_{c,\varepsilon,i}^m$					
cells	L^1	order	L^2	order	L^∞	order
100	6.0E-02	-	6.3E-02	-	2.5E-01	-
200	3.2E-02	0.9	4.0E-02	0.7	2.3E-01	0.1
400	1.7E-02	0.9	2.6E-02	0.6	2.4E-01	0.0
800	8.7E-03	1.0	1.6E-02	0.7	1.7E-01	0.5
1600	4.5E-03	0.9	1.1E-02	0.5	2.2E-01	0.4

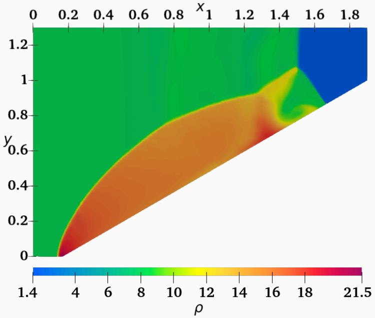
Euler equations: 2D extension

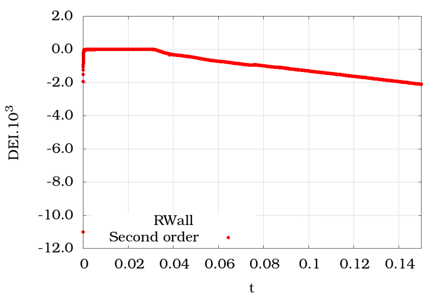
- Isentropic vertex
- $[0, 10]^2, \quad t = 0.1.$
- Unstructered meshes from 400 to 10^5 cells
- Entropy given by $\eta(w) = -\rho \ln(p/\rho^\gamma)$

	h	L^1	order	L^2	order	L^∞	order
	6.2E-01	1.3E-01	-	2.8E-02	-	1.5E-02	-
	3.4E-01	3.4E-02	2.2	8.1E-03	2.0	7.9E-03	1.1
	1.7E-01	1.0E-02	1.8	2.6E-03	1.7	3.6E-03	1.2
	8.9E-02	2.8E-03	2.0	6.7E-04	2.1	7.6E-04	2.4
	4.5E-02	7.8E-04	1.9	2.0E-04	1.8	2.9E-04	1.4

Euler equations: 2D extension

- Shock reflexion
- Unstructered meshes ≈ 115 000 cells
- Entropy given by $\eta(w) = -\rho \ln\left(p/\rho^{\gamma}\right)$





Thank you for your attention