A kinetic model of plasma-probe interaction: theory and numerics

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Séminaire SISMA

Introduction

What is a Langmuir probe ?

A plasma is a gas made of electrically charged particles.

- A Langmuir probe is a spherical or cylindrical metallic measurement device used to study plasmas.
- The probe voltage is varied to be either attractive or repulsive for the electrons and it registers the current.
- It permits to determine the *plasma parameters:* its density, its temperature and its potential.

What is a Langmuir probe ?

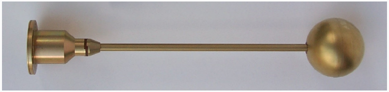


Figure 1: One of the two Langmuir probes from the Swedish institute of Space Physics in Uppsala on board ESA's space vehicle Rosetta

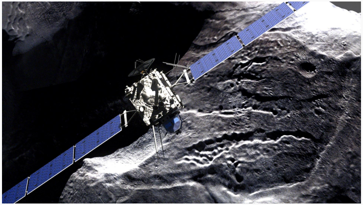


Figure 2: Rosetta in orbit around the 67P/G-C comet

References in physics

The modeling of probe-plasma interaction is a long time discussed problem in plasma physics.

- Smott and Langmuir, The theory of collectors in Gaseous Discharges, 1926.
- Bernstein and Rabinowitz, Theory of electrotatics probes in a low-density plasmas, 1959.
- Allen, Probe theory - the orbital motion approach. Physica Scripta, 1992.
- Laframbroise, Theory of spherical and cylindrical Langmuir probes in collisionless Maxwellian plasmas, 1996.

References in applied mathematics

This problem has not been discussed that much in the mathematical community.

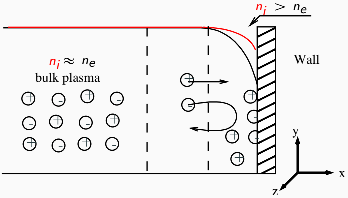
- Raviart and Greengard, A boundary value problem for the stationary Vlasov-Poisson equations: the plane diode, 1990.
- Degond, Raviart and al, The child-Langmuir asymptotics of the Vlasov-Poisson equation for cylindrically of spherically symmetric diode, 1996.

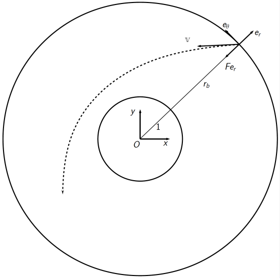
Two difficulties

The mass m_e of an electron is much smaller than the mass m_i of an ion:

- It causes a charge separation in the vicinity of the probe called the Debye sheath.

In non-planar geometry, it is not clear whether a particle reaches the probe.





Model and results

Modeling a cylindrical Langmuir probe

Main assumptions and simplifications:

- The probe is an infinite cylinder of radius 1.

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- The probe is an infinite cylinder of radius 1.
- The plasma is collisionless and unmagnetized (Vlasov-Poisson equations).
- The plasma has reached its permanent regime (steady equations).
- Invariance and symmetries along the probe (polar coordinate).
- Invariance by rotation (radial Poisson equation).
- Invariance by axial symmetry (no ortho-radial current).

Phase-space coordinate system

Particles positions in phase-space in the polar coordinate system write:

$$\begin{cases} \mathbb{x} = (x, y) = r\mathbb{e}_r, \quad r = \sqrt{x^2 + y^2}, \quad \mathbb{e}_{\mathbb{r}} = (\cos\theta, \sin\theta) \\ \mathbb{v} := (v_x, v_y) = v_r\mathbb{e}_r + v_\theta\mathbb{e}_\theta, \quad \mathbb{e}_\theta = (-\sin\theta, \cos\theta), \\ v_r = \mathbb{v} \cdot \mathbb{e}_r, \quad v_\theta = \mathbb{v} \cdot \mathbb{e}_\theta. \end{cases}$$

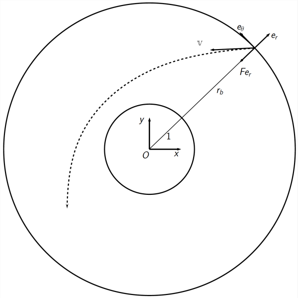

Figure 3: Sketch of a trajectory of a particle into a radial force field coming from the outer ionizing source at $r = r_b$.

The Vlasov-Poisson equations

- Vlasov equation for the ionic density $f_i(r, v_r, v_\theta)$:

$$v_r\,\partial_r f_i - \frac{v_r v_\theta}{r}\partial_{v_\theta} f_i + \left(\frac{v_\theta^2}{r} - \partial_r\phi\right)\,\partial_{v_r} f_i = 0,$$

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- Vlasov equation for the electronic density $f_e(r, v_r, v_\theta)$:

$$v_r\,\partial_r f_e - \frac{v_r v_\theta}{r}\partial_{v_\theta} f_e + \left(\frac{v_\theta^2}{r} + \partial_r\phi\right)\partial_{v_r} f_e = 0,$$

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- Radial Poisson equation for the electrostatic potential $\phi(r)$:

$$-\frac{\lambda^2}{r} \frac{d}{dr} \left(r \frac{d\phi}{dr} \right)(r) = n_i(r) - n_e(r).$$

The domain of computation is $(r, v_r, v_\theta) \in (1, r_b) \times \mathbb{R}^2$. The parameter $\lambda \ll 1$ is the Debye length.

The Vlasov-Poisson equations

The ions and electrons macroscopic charge densities are:

$$n_i(r) := \int_{\mathbb{R}^2} f_i(r, v_r, v_\theta) dv_r dv_\theta, \quad n_e(r) := \int_{\mathbb{R}^2} f_e(r, v_r, v_\theta) dv_r dv_\theta.$$

The ions and electrons radial current densities are:

$$J_i(r) := \int_{\mathbb{R}^2} f_i(r, v_r, v_\theta) v_r dv_r dv_\theta, \quad J_e(r) := \frac{1}{\sqrt{\mu}} \int_{\mathbb{R}^2} f_e(r, v_r, v_\theta) v_r dv_r dv_\theta.$$

where $\mu = m_e/m_i \ll 1$ is the mass ratio.

Boundary conditions

- Incoming particles from the plasma core:

$$\forall v_r < 0 \quad f_i(r_b, v_r, v_\theta) = f_i^b(v_r, v_\theta), \quad f_e(r_b, v_r, v_\theta) = f_e^b(v_r, v_\theta),$$

where $(v_r, v_\theta) \mapsto f_i^b(v_r, v_\theta)$, $(v_r, v_\theta) \mapsto f_e^b(v_r, v_\theta)$ are given functions.

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- Non-emitting Langmuir probe:

$$\forall v_r > 0 \quad f_i(1, v_r, v_\theta) = 0, \quad f_e(1, v_r, v_\theta) = 0.$$

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- Non-emitting Langmuir probe:

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- Boundary datum for the Poisson equation:

$$\phi(1) = \phi_p \in \mathbb{R}, \quad \phi(r_b) = 0.$$

ϕ_p is the probe potential.

An existence theorem

Theorem (B., Godard-Cadillac)

Assume that the incoming particle distributions f_i^b and f_e^b are in L^1 and satisfy the following integrability conditions:

$$\|f\|_{L^1_{v_\theta}(L^\infty_{v_r}(v_r dv_r))} := \int_{\mathbb{R}} \sup_{v_r \in \mathbb{R}} |v_r f(v_r, v_\theta)| dv_\theta < +\infty,$$

$$\|f\|_{L^1_{v_r}(L^\infty_{v_\theta}; |v_r|^{-\gamma} dv_r)} := \int_{\mathbb{R}} \sup_{v_\theta \in \mathbb{R}} |f(v_r, v_\theta)| \frac{dv_r}{|v_r|^\gamma} < +\infty,$$

for some $0 < \gamma < 1$. Then there exists a solution for the Vlasov-Poisson system for the Langmuir probe (weak solution for Vlasov and strong for Poisson). Moreover, the solutions of the Vlasov equations are given by explicit formula depending on ϕ, f_i^b, f_e^b.

A sufficient condition for the integrability conditions:

$$\forall (v_r, v_\theta), \quad |f(v_r, v_\theta)| \leqslant \frac{1}{1 + |v_r| + |v_\theta|^2}.$$

Condition satisfied by Maxwellian distributions.

The radial solutions

Radial solutions are measure valued solutions in the form:

$$f_i(r, v_r, v_\theta) = g_i(r, v_r) \otimes \delta_{v_\theta = 0}, \quad f_e(r, v_r, v_\theta) = g_e(r, v_r) \otimes \delta_{v_\theta = 0}.$$

It is a degenerate case: particles move radially.

Theorem (B., Crestetto, Godard-Cadillac)

Let $\phi_p < 0$, $\lambda > 0$. Assume the incoming radial distribution of particles $g_i^b(v_r)$ and $g_e^b(v_r)$ are in L^1 and $\sup_{v_r \in \mathbb{R}} |v_r g(v_r)| < +\infty$. Assume additionally:

- $g_e^b \in W^{2,1}(\mathbb{R}^-)$ *satisfies some differential inequalities (not written here).*
- *The generalized Bohm condition:*

$$\int_{-\infty}^{0} \frac{g_i^b(w)}{w^2} dw < g_e^b\left(-\sqrt{-2\phi_p}\right)(-2\phi_p)^{-\frac{1}{2}} + \int_{-\infty}^{\sqrt{-2\phi_p}} \frac{dg_e^b}{dw}(-|v|) \frac{dv}{|v|}.$$

- *The neutrality in the plasma core:* $n_i(r_b) = n_e(r_b)$.

The radial solutions

Then, there exists a solution (measure valued solution for the Vlasov equations, classical solution for the Poisson equation) with the following properties:

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- ϕ_λ has the regularity $C^2[1, r_b]$ and it is increasing concave.
- ϕ_λ converges locally uniformly to zero in $(1, r_b]$ as $\lambda \to 0$.
- ϕ verifies the boundary-layer estimate:

$$\frac{\lambda^2}{2}\int_1^{r_b} r\left|\frac{d\phi_\lambda}{dr}(r)\right|^2 dr + \frac{\alpha}{2}\int_1^{r_b} |\phi_\lambda(r)|^2 dr \underset{\lambda \longrightarrow 0}{=} \mathcal{O}(\lambda).$$

where $\alpha > 0$ is a constant independent on λ.

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  where $\alpha > 0$ is a constant independent on λ.
- The plasma is quasi-neutral: $\|n_i - n_e\|_{L^1[1, r_b]} \to 0$ as $\lambda \to 0$.
- ϕ decays exponentially fast: there exists a constant $C > 0$ such that

$$\phi_p \, e^{-C \frac{r-1}{\lambda \sqrt{r_b}}} \leqslant \phi_\lambda(r) \leqslant 0.$$

Some aspects of the analysis

The natural strategy

- Fix the potential $\phi \in W^{2,\infty}[1, r_b]$ and compute explicitly the solutions of the Vlasov equations using the method of characteristics.
- Compute the densities n_i,n_e and study the resulting Poisson equation.

The linear Vlasov equation

- The characteristics of the Vlasov equation with a potential $\psi = \pm\phi$ are given by the solutions of:

$$\begin{cases} \frac{d}{dt} r(t) = v_r(t), \\ \frac{d}{dt} v_r(t) = \frac{v_\theta^2(t)}{r(t)} - \frac{d}{dr}\psi(r(t)), \\ \frac{d}{dt} v_\theta(t) = -\frac{v_r(t) v_\theta(t)}{r(t)}. \end{cases}$$

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- Constants of motion:

$$\frac{d}{dt}\left(\frac{v_r^2(t)}{2} + \frac{v_\theta(t)^2}{2} + \psi(r(t))\right) = 0, \qquad \frac{d}{dt}\left(r(t) v_\theta(t)\right) = 0.$$

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- The characteristics are contained in the level sets defined for $L \in \mathbb{R}$ and $e \in \mathbb{R}$ by:

$$C_{L,e} = \left\{ (r, v_r, v_\theta) \in (1, r_b) \times \mathbb{R}^2 \ : \ \frac{v_r^2}{2} + \frac{v_\theta^2}{2} + \psi(r) = e \quad \text{and } r v_\theta = L \right\}.$$

Phase space study

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- For each $L \in \mathbb{R}$, study the phase space (r, v_r) by looking at the level sets of the function:

$$(r, v_r) \mapsto \frac{v_r^2}{2} + \mathcal{U}_L[\psi](r).$$

- The maximum value

$$\overline{\mathcal{U}_L[\psi]} = \max_{r \in [1, r_b]} \mathcal{U}_L[\psi](r)$$

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$$\overline{\mathcal{U}_L[\psi]} = \max_{r \in [1, r_b]} \mathcal{U}_L[\psi](r)$$

is a global potential barrier.

Phase space study

- Cover the phase space (r, v_r) with the curves of equation: $\frac{v_r^2}{2} = e - \mathcal{U}_L[\psi](r)$.

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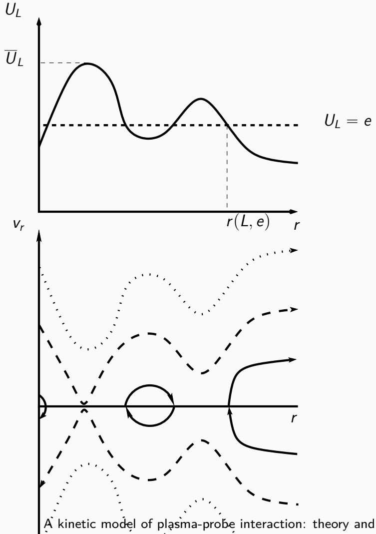
- Cover the phase space (r, v_r) with the curves of equation: $\frac{v_r^2}{2} = e - \mathcal{U}_L[\psi](r)$.
- Study the barrier position when $e < \overline{\mathcal{U}_L[\psi]}$:

$$r(L, e) := \min\{a \in [1, r_b] \ : \ \forall r \in [a, r_b], \quad \mathcal{U}_L[\psi](r) \leqslant e\}.$$



Phase space study

- Decomposition of the phase space:

$$D_b^1[\psi](L) := \left\{ (r, v_r) \in (1, r_b) \times \mathbb{R} \ : \ v_r < -\sqrt{2(\overline{\mathcal{U}_L[\psi]} - \mathcal{U}_L[\psi](r))} \right\},$$

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- The solution of the Vlasov equation is constant on the characteristics, it is natural to define:

$$f(r, v_r, v_\theta) = \begin{cases} f^b(-\sqrt{v_r^2 + 2(\mathcal{U}_L[\psi](r) - \mathcal{U}_L[\psi](r_b))}, \frac{rv_\theta}{r_b}) \text{ if } (r, v_r) \in D_b[\psi](L),\; L = rv_\theta, \\ 0 \text{ otherwise.} \end{cases}$$

It defines a weak solution to the Vlasov equation for a potential $\psi = \pm\phi$.

Computation of the macroscopic densities and currents

- Define $\tilde{\rho}[\psi] := \inf\{a \in [1, r_b] \ : \ \text{for a.e } r \in [a, r_b], \quad \psi(r) \leqslant 0\}$.

Computation of the macroscopic densities and currents

- Define $\tilde{\rho}[\psi] := \inf\{a \in [1, r_b] \ : \ \text{for a.e } r \in [a, r_b], \quad \psi(r) \leqslant 0\}$.

Then

$$r(L, e) = \tilde{\rho}[\mathcal{U}_L[\psi] - e].$$

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- Define:

$$\begin{array}{rccc} \beta: & \mathbb{R} \times [1, r_b] \times \mathbb{R} & \longrightarrow & \mathbb{R} \\ & (\nu, r, L) & \longmapsto & 2\nu + L^2\left(\dfrac{1}{r^2} - \dfrac{1}{r_b^2}\right). \end{array}$$

- Define:

$$\begin{array}{rccc} \Gamma: & \mathbb{R} \times [1, r_b] \times \mathbb{R} \times \mathbb{R} & \longrightarrow & \mathbb{R} \\ & (\nu, r, w, L) & \longmapsto & \begin{cases} \dfrac{(w)_-}{\sqrt{w^2 - \beta(\nu, r, L)}} & \text{if } w^2 > \beta(\nu, r, L), \\ 0 & \text{otherwise.} \end{cases} \end{array}$$

Computation of the macroscopic densities and currents

Proposition

The macroscopic density and current associated with a potential $\psi = \pm\phi$ are given by:

$$n[\psi](r) = \frac{1}{r} g[\psi](\psi(r), r)$$

$$g[\psi](\nu, r) = \int_{\mathbb{R}^2} \Gamma(\nu, r, w, L)\, f^b\left(w, \frac{L}{r_b}\right)\left(1 + \mathbb{1}_{w^2 + \frac{L^2}{r_b^2} < 2\overline{\mathcal{U}_L}[\psi]}\right)\mathbb{1}_{r \geqslant \tilde{\rho}\left[\psi + \frac{L^2}{2}\left(\frac{1}{\bullet^2} - \frac{1}{r_b^2}\right) - w^2\right]}\, dw\, dL,$$

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$$J[\psi](r) = \frac{2}{r}\int_{L=0}^{L=+\infty}\int_{-\infty}^{-\sqrt{2(\overline{\mathcal{U}_L}[\psi]] - \mathcal{U}_L[\psi](r_b))}} f^b\left(w; \frac{L}{r_b}\right) w\, dw\, dL.$$

The quantities $\overline{\mathcal{U}_L}[\psi]$ and $\tilde{\rho}$ are non-local.

Existence of a solution

- We are interested in solving the Poisson problem:

$$\begin{cases} -\lambda^2 \frac{d}{dr}\left(r \frac{d\phi}{dr} \right)(r) = g[\phi](\phi(r), r) - g[-\phi](-\phi(r), r), \\ \phi(1) = \phi_p \quad \phi(r_b) = 0. \end{cases}$$

- The bracket $[\phi]$ encodes the non-locality.
- The source term in the Poisson equation thus is non-linear and non-local.

Existence of a solution: a fixed-point method

The strategy:

- Fix the non-local terms.

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Existence of a solution: a fixed-point method

The strategy:

- Fix the non-local terms.
- Solve the local semi-linear Poisson problem.
- Establish compactness.
- Pass to the limit to conclude.

Let $\phi^n \in W^{2,\infty}[1, r_b]$ such that $\phi^n(1) = \phi_p$ and $\phi^n(r_b) = 0$. Solve for ϕ^{n+1} :

$$\begin{cases} -\lambda^2 \frac{d}{dr}\left(r \frac{d\phi^{n+1}}{dr}\right)(r) = g[\phi^n](\phi(r)^{n+1}, r) - g[-\phi^n](-\phi(r)^{n+1}, r), \\ \phi^{n+1}(1) = \phi_p \quad \phi^{n+1}(r_b) = 0. \end{cases}$$

Existence of a solution: compactness

A key estimate to extract a converging subsequence is given by the following

Lemma (Functions g_i and g_e are finite)

Let $f : \mathbb{R}^2 \to \mathbb{R}$ *measurable and let* $p \in [1,2)$*. Then,*

$$\sup_{\nu\in\mathbb{R}} \sup_{r\in[1,r_b]} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{|w|^p}{\left|w^2 - L^2\left(\frac{1}{r^2} - \frac{1}{r_b^2}\right) - 2\nu\right|^{\frac{p}{2}}} \left|f(w,L)\right| dw\, dL \leqslant 2\|f\|_{L^1} + \frac{4}{2-p}\|f\|_{L^1_L(L^\infty_w(w\,dw))};$$

The non linear source term $g[\phi^n](\phi(r)^{n+1}, r) - g[-\phi^n](-\phi(r)^{n+1}, r)$ is uniformly bounded in L^∞. We get that the sequence (ϕ^n) is bounded in $W^{2,\infty}$. This implies compactness by Rellich-Kondrachov theorem. In particular ϕ^n converges (up to a sub seq) uniformly to some ϕ.

Existence of a solution: passing to the limit

How to pass to the limit in the quantity:

$$\tilde{\rho}[\psi - e] := \inf\{a \in [1, r_b] \; \forall r \geqslant a \quad \psi(r) \leqslant e\}.$$

It is in general not continuous with respect to ψ for the L^∞ topology. The problem is at strict local maxima of ψ.

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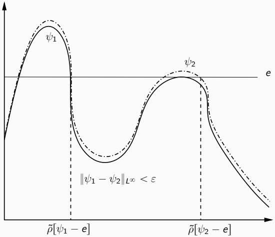

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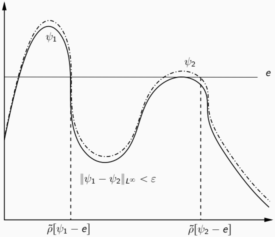

Lemma (Convergence property for $\widetilde{\rho}$)

Let (ϕ_n) be a sequence of continuous functions that is uniformly converging towards ϕ. Then for almost every $e \in \mathbb{R}$,

$$\widetilde{\rho}[\phi_n - e] \longrightarrow \widetilde{\rho}[\phi - e].$$

Enough to pass to the limit since $\tilde{\rho}$ only appears under an integral. The final ingredient to conclude is an equicontinuity estimate of the non linear source term of the Poisson equation.

Numerics

Gradient descent based iterative method: continuous setting

Iterative method:

$$-\lambda^2 \frac{d}{dr}\left(r\frac{d\phi^{n+1}}{dr}\right)(r) = g[\phi^n](\phi(r)^{n+1}, r) - g[-\phi^n](-\phi(r)^{n+1}, r)$$

- If ϕ^n is given, define $G[\phi^n](\nu, r) = \int_0^\nu g[\phi^n](\nu', r)d\nu'$.

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- If ϕ^n is given, define $G[\phi^n](\nu, r) = \int_0^\nu g[\phi^n](\nu', r)d\nu'$.
- Consider the energy functional:

$$J[\phi^n](\psi) = \int_1^{r_b} \frac{\lambda^2}{2} r \left|\frac{d\psi}{dr}(r)\right|^2 + G[-\phi^n](-\psi(r), r) - G[\phi^n](\psi(r), r)dr.$$

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- Compute iteratively:

$$\phi^{n+1} = \phi^n - \rho\nabla J[\phi^n](\phi^n), \rho > 0.$$

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  The gradient $\nabla J[\phi^n](\phi^n) \in V$ is the unique Riesz-representation of the Fréchet differential of J at ϕ^n for a chosen inner product $(\cdot\,;\,\cdot)$:

$$(\nabla_h J[\phi_h^n](\phi_h^n); \varphi) = dJ[\phi_h^n](\phi_h^n)(\varphi) \quad \forall \varphi \in V_h^0$$

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- At convergence, it solves the non-linear and non-local Poisson problem.

Gradient descent based iterative method: discrete setting

Uniform mesh of the interval $[1, r_b]$ of size $h = (r_b - 1)/(N + 1)$
$\phi_h^n \in V_h :=$ continuous and piecewise affine functions on $[1, r_b]$.

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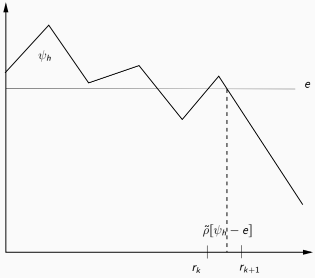

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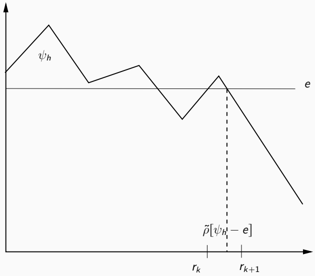

- The indicator $\mathbb{1}_{w^2 + \frac{L^2}{r_b^2} < 2\overline{\mathcal{U}_L}[\psi]}$ is regularized because oscillations may appear if high order numerical integration is used.

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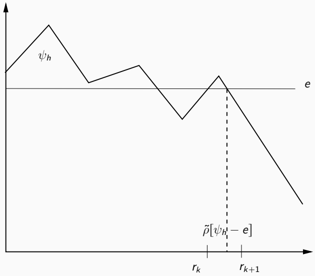

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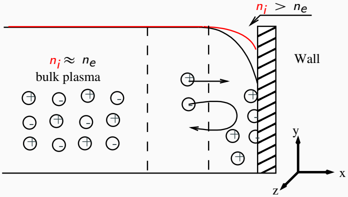
Radial solutions: the 1D sheath problem



The one dimensional sheath problem is a classic of the plasma physics [1]. We have done a mathematical analysis to validate the code. Boundary conditions are in the form:

$$f_i^b(v_r, v_\theta) = g_i^b(v_r) \otimes \delta_{v_\theta=0}, \quad f_e^b(v_r, v_\theta) = g_e^b(v_r) \otimes \delta_{v_\theta=0},$$

with

$$g_i^b(v_r) = \frac{v_r^2}{\sqrt{2\pi}} e^{-\frac{(v_r-u_i)^2}{2}}, \; u_i = -2.0 \quad g_e^b(v_r) = \frac{n^b}{\sqrt{2\pi}} e^{-\frac{v_r^2}{2}}.$$

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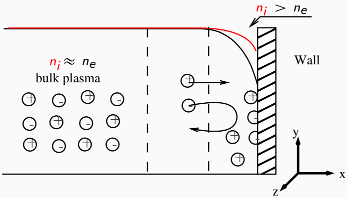
Radial solutions: the 1D sheath problem



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- n_b is fixed a priori to ensure $n_i(r_b) = n_e(r_b)$.
- The Bohm condition $\int_{-\infty}^0 g_i^b(v_r) v_r^{-2} dv_r < 1$ is verified.

Numerical parameters: $r_b = 3, \lambda = 0.1, N = 200$. The gradient algorithm is stopped when $\|\nabla_h J[\phi_h^n](\phi_h^n)\|_{L^\infty} < 10^{-8}$.

[1] K-U Riemann, The Bohm criterion and Sheath formation, Journal of Physics D: Applied Physics, 1991

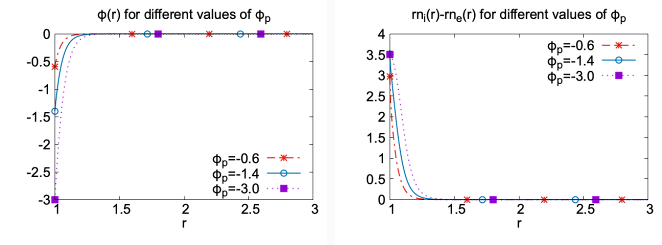

Figure 4: Radial case with satisfied Bohm condition: potential $\phi(r)$ (left) and density difference $rn_i(r) - rn_e(r)$ (right) for ϕ_p varying from -0.6 to -3.

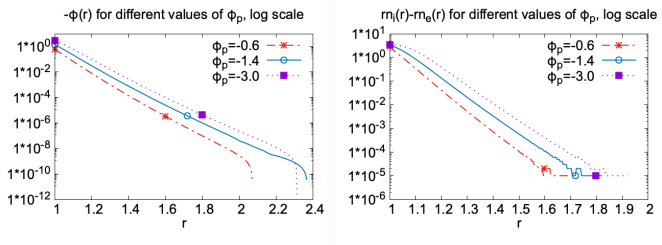

Figure 5: Radial case with satisfied Bohm condition: potential $\phi(r)$ (left) and density difference $rn_i(r) - rn_e(r)$ (right) for ϕ_p varying from -0.6 to -3 in semi-log scale.

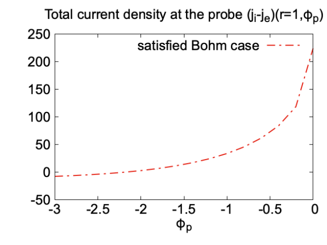

Figure 6: Radial case: total current density at the probe $(j_i - j_e)(r = 1, \phi_p)$ as a function of the probe potential.

The radial current is a monotone increasing function of ϕ_p.

Two dimensional solutions : perturbation of the radial setting

It is a prospective test case. This is motivated by the work of Laframbroise [2]. Boundary conditions are in the form:

$$f_i^b(v_r, v_\theta) = \frac{1}{\sqrt{2\pi}} e^{-\frac{v_r^2}{2}} \otimes \mathcal{M}_T(v_\theta), \quad f_e^b(v_r, v_\theta) = \frac{n_b}{\sqrt{2\pi}} e^{-\frac{v_r^2}{2}} \otimes \mathcal{M}_T(v_\theta)$$

where

$$\mathcal{M}_T(v_\theta) = \frac{1}{\sqrt{2\pi T}} e^{-\frac{v_\theta^2}{2T}}, \quad T > 0.$$

- n_b is fixed a priori to ensure $n_i(r_b) = n_e(r_b)$.
- Two cases are simulated : $T = 0.1$ and $T = 0.05$.

Numerical parameters: $r_b = 3, \lambda = 0.1, N = 200$. The gradient algorithm is stopped when $\|\nabla_h J[\phi_h^n](\phi_h^n)\|_{L^\infty} < 10^{-4}$.

[2] Laframbroise, Theory of spherical and cylindrical Langmuir probes in collisionless Maxwellian plasmas, 1996.

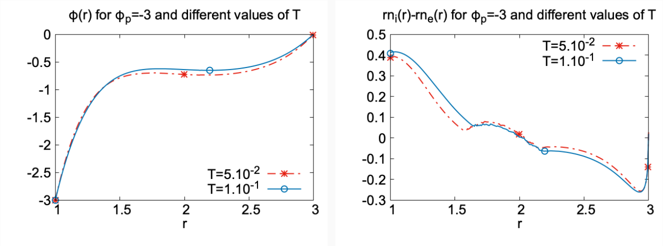

Figure 7: Maxwellian case: potential $\phi(r)$ (left) and density difference $rn_i(r) - rn_e(r)$ (right) for $\phi_p = -3$ and two values of T: 0.05 and 0.1.

There seems to be a loss of monotony in the potential. It may yield the existence of unpopulated orbits in the phase space. The potential does not seem to connect smoothly with the plasma core $r > r_b$.

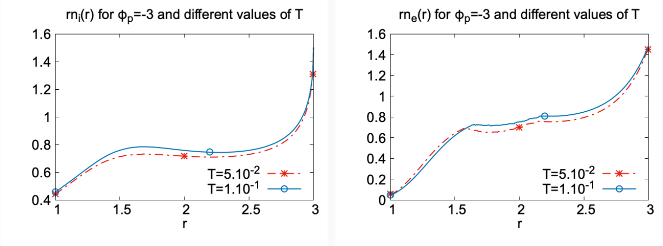

Figure 8: Maxwellian case: potential $\phi(r)$ (left) and density difference $rn_i(r) - rn_e(r)$ (right) for $\phi_p = -3$ and two values of T: 0.05 and 0.1.

There is a clear loss of monotony in the ionic and electronic densities. It seems that unpopulated orbits do exist: in the range $r \in [1.7, 2.7]$ for the ions and in the range $r \in [1.5, 2.0]$ for the electrons.

Ionic phase space: $(r, v_r, L = 0)$

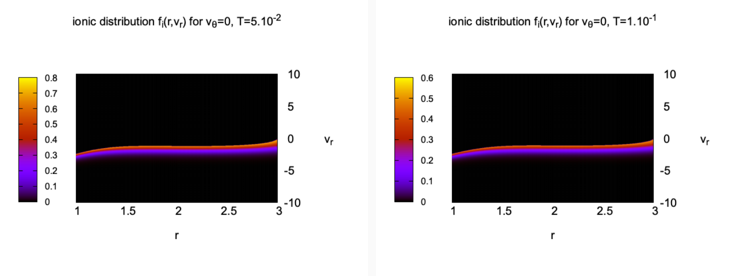

Figure 9: Maxwellian case: ionic distribution function $f_i(r, v_r)$ for $T = 0.05$ (left), $T = 0.1$ (right).

Two dimensional solutions: ionic phase space

Ionic phase space: $(r, v_r, L \neq 0)$

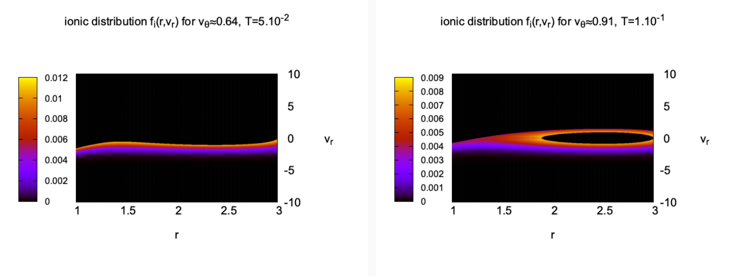

Figure 10: Maxwellian case: ionic distribution function $f_i(r, v_r)$ for $T = 0.05$ (left), $T = 0.1$ (right)

The effective potential is $U_L(r) = +\phi(r) + \frac{L}{2r^2}$.

Electronic phase space: $(r, v_r, L = 0)$

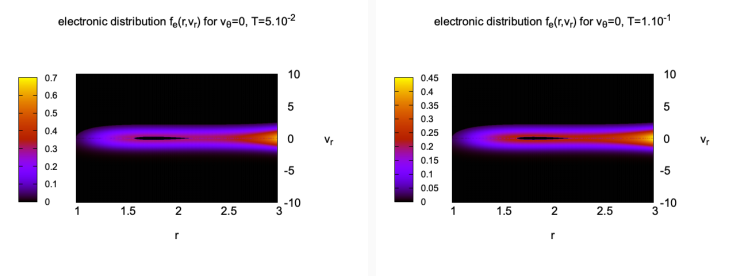

Figure 11: Maxwellian case: ionic distribution function $f_e(r, v_r)$ for $T = 0.05$ (left), $T = 0.1$ (right).

Electronic phase space: $(r, v_r, L \neq 0)$

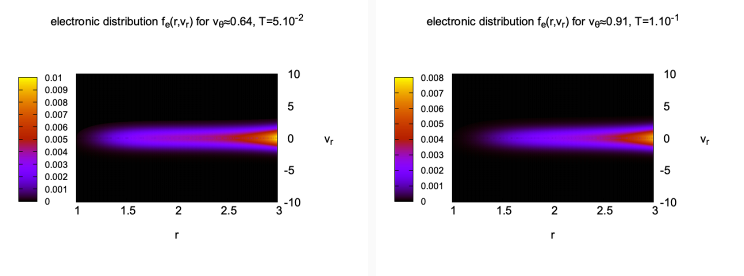

Figure 12: Maxwellian case: ionic distribution function $f_e(r, v_r)$ for $T = 0.05$ (left), $T = 0.1$ (right), and three increasing values of v_θ from top to bottom.

The effective potential is $U_L(r) = -\phi(r) + \frac{L}{2r^2}$.

Two dimensional solutions: the probe characteristic (I-V curve)

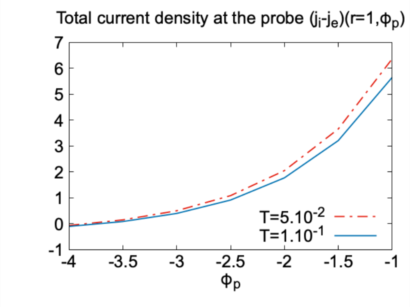

Figure 13: Maxwellian case: total current density at the probe $(j_i - j_e)(r = 1, \phi_p)$ as a function of the probe potential.

Still seems to be monotone for the range of parameter we used. Not proven and not clear !

Two dimensional solutions : ionic trajectories

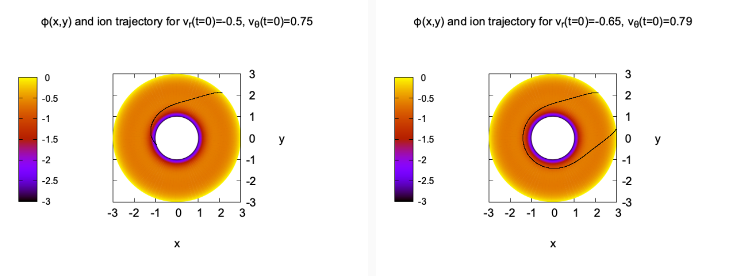

Figure 14: Maxwellian case: ϕ and ionic trajectories.

Two dimensional solutions : electronic trajectories

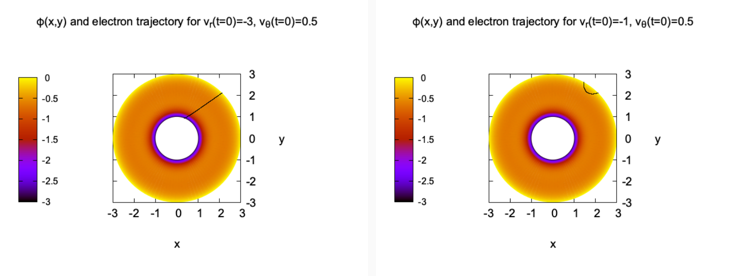

Figure 15: Maxwellian case: ϕ and electronic trajectories.

Conclusion

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[3] Laframbroise, Theory of spherical and cylindrical Langmuir probes in collisionless Maxwellian plasmas, 1996.

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- We obtained qualitative description and quantitative estimates of the solutions in the radial setting.
- We proposed a numerical method to compute the solutions which is able to capture unpopulated orbits.
- Comparison with the results of Laframboise [3] needs more numerical investigation.
- Dissemination:
  - "Existence of solutions for a bi species kinetic model of a cylindrical Langmuir probe, M.Badsi and L. Godard-Cadillac , in Communications in Mathematical Sciences, 2022"
  - "Variational radial solutions and numerical simulations for a kinetic model of a cylindrical Langmuir probe, A. Crestteto, M.Badsi and L. Godard-Cadillac, preprint HAL, 2022."

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Thank you for paying attention.