Two approximations of the Euler equations using a non conservative formulation

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Joint work with K. Ivanova (ISL, Davos)
Acknowledge discussion with W. Barsukow (UZH then MPI, München, then CNRS, Bordeaux)

Questions:

- What does conservation mean?
- What are the real requirement of numerical schemes (for compressible problems)?
- If we meet this, maybe we can get additional freedom to fullfill additional, and contradictory, properties.

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Overview

Introduction-Problem statement

In $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$:

$$\frac{\partial \mathbf{u}}{\partial t} + \text{ div } \mathbf{f}(\mathbf{u}) = 0 \qquad + \text{initial and boundary conditions}$$

$$\mathbf{u} = \begin{pmatrix} \rho \\ \rho \mathbf{v} \\ E \end{pmatrix}, \mathbf{f} = \begin{pmatrix} \rho \mathbf{v} \\ \rho \mathbf{v} \otimes \mathbf{v} + p\mathrm{Id} \\ \mathbf{v}(E + p) \end{pmatrix}$$

with equation of state $p = p(\rho, e)$.

"non conservative version":

$$\frac{\partial \rho}{\partial t} + \text{ div } \rho \mathbf{v} = 0$$

$$\frac{\partial \mathbf{v}}{\partial t} + [\mathbf{v} \cdot \nabla] \mathbf{v} + \frac{\nabla p}{\rho} = 0$$

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Purpose of the talk: describe (and analyse) 2 ways of discretizing the PDE using directly the "non conservative" formulation: staggered grid and "Active flux" type

Overview

Staggered grid approach

- $\Omega = \cup K$, K simplex and conformal mesh
- Approximation: thermodynamics/kinematics

$$\rho, e, p \in \oplus_K \mathbb{P}^r(K) \subset L^2(\Omega),$$

$$\mathbf{v} \in \big(\oplus_K \mathbb{P}^p(K)\big)^d \qquad \Big(\cap\, C^0(\Omega)\Big)$$

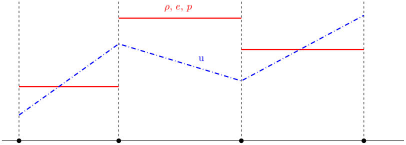
Approximation 1D-to make things simpler

linear velocity, piece-wise constant thermodynamic

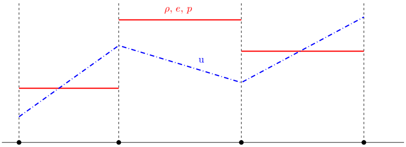
$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0$$

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} = 0$$

$$\frac{\partial e}{\partial t} + u \frac{\partial e}{\partial x} + (e + p) \frac{\partial u}{\partial x} = 0$$



Density



Use . . . a numerical flux $\hat{\mathbf{f}}$.
Riemann problem: discontinuous ρ, p, continuous velocity

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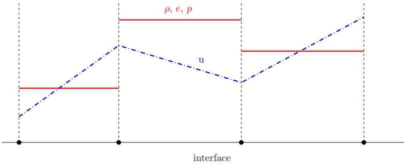

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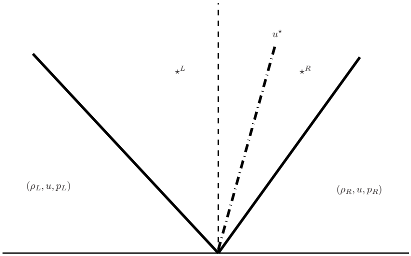

Godunov: star states defined

velocity

$w \in \oplus_K \mathbb{P}^r(K)$, here $r = 1$
"weak" form

$$\int_\Omega w \rho \frac{\partial \mathbf{u}}{\partial t}\ d\mathbf{x} + \int_\Omega \rho [\mathbf{u} \cdot \nabla] \mathbf{u} - \int_\Omega p \nabla w\ d\mathbf{x} = 0$$

Choosing test function (and specialise to $r = 1$ for simplicity: $w = \varphi_i$

$$\int_{x_{i-1}}^{x_{i+1}} \varphi_i \frac{\partial \mathbf{u}}{\partial t} + \overleftarrow{\Phi}_{i+1/2} + \overrightarrow{\Phi}_{i-1/2} = 0$$

with

$$\hat{\rho}_i \overleftarrow{\Phi}_{i+1/2} = \int_{x_i}^{x_{i+1}} \varphi_i \rho \mathbf{u} \frac{\partial \mathbf{u}}{\partial x}\ d\mathbf{x} - \int_{x_i}^{x_{i+1}} p \frac{\partial \varphi_i}{\partial x}\ d\mathbf{x} + (-)$$

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and

$$\int_{x_{i-1}}^{x_{i+1}} \varphi_i \frac{\partial \mathbf{u}}{\partial t}\ d\mathbf{x} \approx \frac{\int_{x_{i-1}}^{x_{i+1}} \varphi_i \rho \frac{\partial \mathbf{u}}{\partial t}\ d\mathbf{x}}{\hat{\rho}_i} \approx (x_{i+1/2} - x_{i-1/2}) \frac{\partial \mathbf{u}_i}{\partial t}$$

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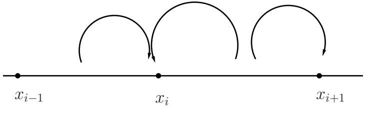

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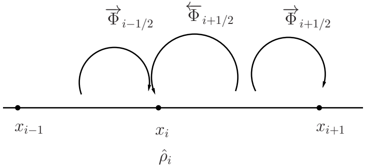

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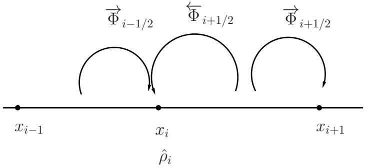

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Internal energy

"weak" form: $w \in L^2(K)$,

$$-\int_K w \frac{\partial e}{\partial t}\ d\mathbf{x} = \int_K w \big(\mathbf{u} \cdot \nabla e + (e+p)\ \mathrm{div}\ \mathbf{u}\big)\ d\mathbf{x}$$

For piecewise constant:

$$\int_{x_i}^{x_{i+1}} \varphi \frac{\partial e}{\partial t}\ dx + \int_{x_i}^{x_{i+1}} \big(-e\ \mathbf{u} \cdot \nabla \varphi + p\ \mathrm{div}\ \mathbf{u}\big)\ dx + \Big(\varphi(x_{i+1}) e^{\star}_{i+1} u_{i+1} - \varphi(x_i) e^{\star}_i u_i\Big) = 0$$

and (here) $\varphi = 1$

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&= \int_K \left[\left(\ \text{div}\ (w\ e\ \mathbf{u}) - w\ e\ \text{div}\ \mathbf{u} - e\ \mathbf{u}\cdot\nabla w\right) + (e+p)\ \text{div}\ \mathbf{u}\right] d\mathbf{x} \\
&= \int_K \left[\left(\ \text{div}\ (w\ e\ \mathbf{u}) - e\ \mathbf{u}\cdot\nabla w\right) + p\ \text{div}\ \mathbf{u}\right] d\mathbf{x} \\
&= \int_{\partial K} w\ e\ \mathbf{u}\cdot\mathbf{n} d\gamma + \int_K \left(-\ e\ \mathbf{u}\cdot\nabla w + p\ \text{div}\ \mathbf{u}\right) d\mathbf{x}
\end{aligned}
$$

For piecewise constant:

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\int_{x_i}^{x_{i+1}} \varphi \frac{\partial e}{\partial t}\, dx + \int_{x_i}^{x_{i+1}} \left(-\ e\ \mathbf{u}\cdot\nabla\varphi + p\ \text{div}\ \mathbf{u}\right) dx + \left(\varphi(x_{i+1}) e^\star_{i+1} u_{i+1} - \varphi(x_i) e^\star_i\, u_i\right) = 0
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Of course this cannot work...!

So far, we have (with Euler forward discretisation in time)

$$|C_K|\left(\rho_K^{n+1}-\rho_K^n\right)+\Phi_K^\rho=0$$
$$|C_i|(u_i^{n+1}-u_i^n)+\overleftarrow{\Phi}^u_{i+1/2}+\overrightarrow{\Phi}^u_{i-1/2}=0$$
$$|C_K|\left(e_K^{n+1}-e_K^n\right)+\Phi_K^e=0$$

σ_V, σ_E kinetic/thermodynamics dofs

$$|C_{\sigma_E}|\left(\rho_{\sigma_E}^{n+1}-\rho_{\sigma_E}^n\right)+\Phi_{\sigma_E}^\rho=0$$
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\end{aligned}
$$

How can we guaranty local conservation??
Related work by Latché, Herbin et al. in FV context

Back to basics

Example of momentum

v: $C_0^1(\mathbb{R}^d \times \mathbb{R}^+)$ test function, Evaluate/estimate

$$\int_{\mathbb{R}^d} v\left(\rho^{n+1}(x)\mathbf{u}^{n+1}(x) - \rho^n(x)\mathbf{u}^n(x)\right)\, d\mathbf{x}$$

Questions: What are ρ^k and $\mathbf{u}^k$ for $k = n, n+1$?

- Density: approximate in $\oplus_K \mathbb{P}^d(K) \subset L^2$.
  Dofs σ_E,

$$\rho(x) = \sum_K \sum_{\sigma_E \in K} \rho_{\sigma_E} \varphi_{\sigma_E}, \quad p(x) = \sum_K \sum_{\sigma_E \in K} p_{\sigma_E} \varphi_{\sigma_E}, \quad e(x) = \sum_K \sum_{\sigma_E \in K} e_{\sigma_E} \varphi_{\sigma_E}.$$

- Velocity in $\oplus_K \mathbb{P}^d(K) \cap C_0$:

$$\mathbf{u} = \sum_K \sum_{\sigma_V \in K} \mathbf{u}_{\sigma_V} \psi_{\sigma_V} \in C_0.$$

- Localise $\mathcal{V}$:

$$\mathcal{V} \approx \sum_K \mathcal{V}_K \chi_K, \qquad \mathcal{V}_K = v(\text{centroid of } K).$$

- Write:

$$\rho^{n+1}\mathbf{u}^{n+1} - \rho^n\mathbf{u}^n = \rho^{n+1}\left(\mathbf{u}^{n+1} - \mathbf{u}^n\right) + \mathbf{u}^n\left(\rho^{n+1} - \rho^n\right).$$

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Example of momentum

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$$\mathbf{u} = \sum_K \sum_{\sigma_V \in K} \mathbf{u}_{\sigma_V}\psi_{\sigma_V} \in C_0.$$

- Localise $\mathcal{V}$:

$$\mathcal{V} \approx \sum_K \mathcal{V}_K \chi_K, \qquad \mathcal{V}_K = v(\text{centroid of } K).$$

- Write:

$$\rho^{n+1}\mathbf{u}^{n+1} - \rho^n\mathbf{u}^n = \rho^{n+1}\left(\mathbf{u}^{n+1} - \mathbf{u}^n\right) + \mathbf{u}^n\left(\rho^{n+1} - \rho^n\right).$$

Back to basics

Example of momentum

v: $C_0^1(\mathbb{R}^d \times \mathbb{R}^+)$ test function, Evaluate/estimate

$$\int_{\mathbb{R}^d} v\left(\rho^{n+1}(x)\mathbf{u}^{n+1}(x) - \rho^n(x)\mathbf{u}^n(x)\right)\, d\mathbf{x}$$

Questions: What are ρ^k and $\mathbf{u}^k$ for $k = n, n+1$?

- Density: approximate in $\oplus_K \mathbb{P}^d(K) \subset L^2$.
  Dofs σ_E,

$$\rho(x) = \sum_K \sum_{\sigma_E \in K} \rho_{\sigma_E}\varphi_{\sigma_E}, \quad p(x) = \sum_K \sum_{\sigma_E \in K} p_{\sigma_E}\varphi_{\sigma_E}, \quad e(x) = \sum_K \sum_{\sigma_E \in K} e_{\sigma_E}\varphi_{\sigma_E}.$$

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$$= \sum_K \mathcal{V}_K \Big\{ \sum_{\sigma_V \in K} (\mathbf{u}^{n+1}_{\sigma_V} - \mathbf{u}^n_{\sigma_V}) \int_K \rho^{n+1}\psi_{\sigma_V} + \sum_{\sigma_E} (\rho^{n+1}_{\sigma_E} - \sigma^n_{\sigma_E}) \int_K \varphi_{\sigma_E} \mathbf{u}^n(x)\, d\mathbf{x} \Big\}$$

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$$+ \sum_{\sigma_E} |C_{\sigma_E}| (\rho^{n+1}_{\sigma_E} - \sigma^n_{\sigma_E}) \underbrace{\frac{\int_K \varphi_{\sigma_E} \mathbf{u}^n(x)\, d\mathbf{x}}{|C_{\sigma_E}|}}_{\omega^{\rho,K}_{\sigma_E}} \Big\}$$

- Use the scheme written as

$$|C_{\sigma_V}| (\mathbf{u}^{n+1}_{\sigma_V} - \mathbf{u}^n_{\sigma_V}) + \Delta t \sum_{K, \sigma_V \in K} \Phi^K_{\sigma_V} = 0, \quad |C_{\sigma_E}| (\rho^{n+1}_{\sigma_E} - \rho^n_{\sigma_E}) + \Phi^{K,\rho}_{\sigma_E} = 0$$

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- Some algebra...

Back to the basics

...and get:

$$\int_{\mathbb{R}^d} v\left(\mathbf{m}(x)^{n+1} - \mathbf{m}(x)^n\right)\, d\mathbf{x} + \Delta t \sum_K \mathcal{V}_K \left\{ \sum_{\sigma_V \in K} \omega^{\mathbf{u}}_{\sigma_E} \Phi^K_{\sigma_V} + \sum_{\sigma_E \in K} \omega^{\rho,K}_{\sigma_E} \Phi^{\rho}_{\sigma_E} \right\} + \text{terms} = 0$$

with

$$\omega^{\mathbf{u}}_{\sigma_E} = \frac{\sum_{K,\sigma_V \in K} \int_K \rho^{n+1} \psi_{\sigma_V}}{|C_{\sigma_E}|}, \omega^{\rho,K}_{\sigma_E} = \frac{\int_K \varphi_{\sigma_E} \mathbf{u}^n(x)\, d\mathbf{x}}{|C_{\sigma_E}|}$$

Suffisant condition

- Momentum

$$\sum_{\sigma_V \in K} \omega^{\mathbf{u}}_{\sigma_E} \Phi^K_{\sigma_V} + \sum_{\sigma_E \in K} \omega^{\rho,K}_{\sigma_E} \Phi^{\rho}_{\sigma_E} = \int_{\partial K} \mathbf{f_m} \cdot \mathbf{n}\, d\gamma$$

- Energy

$$\sum_{\sigma_E \in K} \Phi^e_{\sigma_E,K} + \sum_{\sigma_V \in K} \theta^{\mathbf{m}}_{\sigma_V} \cdot \Phi^{\mathbf{u}}_{\sigma_V,K} + \frac{1}{2} \sum_{\sigma_E} \theta^{q^2,K}_{\sigma_V} \Phi_{\sigma_E} = \int_{\partial K} \mathbf{f}^E(U) \cdot \mathbf{n}\, d\gamma$$

with

$$\theta^{\mathbf{m}}_{\sigma_V} = \frac{\sum\limits_{K,\sigma_V \in K} \int_K \widetilde{\mathbf{m}} \varphi_{\sigma_V}\, dx}{|C_{\sigma_V}|}, \qquad \theta^{q^2,K}_{\sigma_V} = \frac{\int_K \widetilde{q^2} \varphi_{\sigma_E}\, dx}{|C_{\sigma_E}|}$$

- Sum: "standard" arguments, Weak BV estimates

Back to the basics

... and get:

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$$\sum_{\sigma_V \in K} \omega^{\mathbf{u}}_{\sigma_E} \Phi^K_{\sigma_V} + \sum_{\sigma_E \in K} \omega^{\rho,K}_{\sigma_E} \Phi^{\rho}_{\sigma_E} = \int_{\partial K} \mathbf{f_m} \cdot \mathbf{n}\ d\gamma$$

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- Sum: "standard" arguments, Weak BV estimates

Of course these conditions are not met in general but one can force them:

1. Compute ρ^{n+1}, keep ρ^n, $\mathbf{u}^n$, e^n
2. Modify the local update of the velocities so that $(\Phi^K_{\sigma_V} \to \Phi^K_{\sigma_V} + r^K_V)$

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   holds true, and update the velocity: we have ρ^{n+1}, $\mathbf{u}^{n+1}$, keep ρ^n, $\mathbf{u}^n$, e^n

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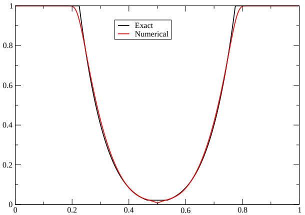
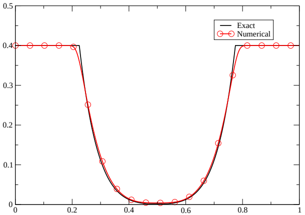
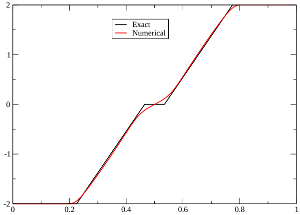
$$(\rho_0, u_0, p_0) = \begin{cases} (1.0, -2.0, 0.4), & \textit{if} \quad 0.0 \leq x < 0.5, \\ (1.0, 2.0, 0.4), & \textit{if} \quad 0.5 < x < 1, \end{cases} \tag{1}$$



(a)



(b)



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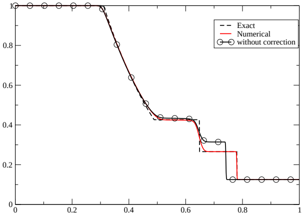
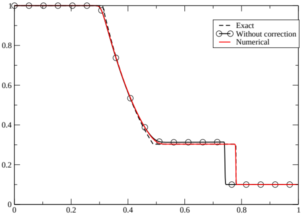

Exact
Numerical
without correction

(a)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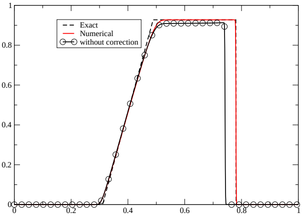
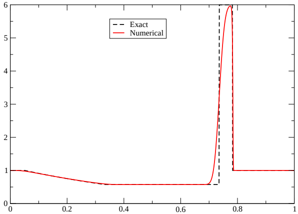
(b)


Exact
Numerical
without correction

$$(\rho_0, u_0, p_0) = \begin{cases} (1.0, 0.0, 1000.0), & \text{if} \quad x < 0.5, \\ (1.0, 0.0, 0.01), & \text{if} \quad x > 0.5, \end{cases} \tag{2}$$



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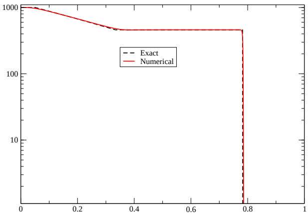

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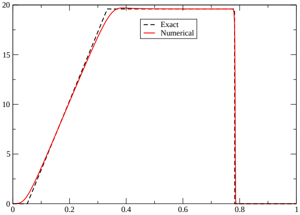

Overview

Another point of view

Question/facts/Motivations

- Can we use at the same time the conservative and non conservative version of the same model ?
- More cryptic: Initial motivation by Phil Roe and simplify time stepping of RDS
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Idea

Given a problem (say compressible)

- Consider 2 versions

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{u})}{\partial x} = 0 \qquad \text{and } \frac{\partial \mathbf{v}}{\partial t} + A\frac{\partial \mathbf{v}}{\partial x} = 0$$

  with
  - $\mathbf{u} = (\rho, \rho u, E)$, and $\mathbf{v} = \mathbf{u} = \Psi(\mathbf{u})$, sof $\Psi = \text{Id}$ is C^1 at least. This is the AF version, in a way.
  - $\mathbf{u} = (\rho, \rho u, E)$, and $\mathbf{v} = (\rho, u, p)$, $\mathbf{u} = \Psi(\mathbf{v})$ is C^1 at least.
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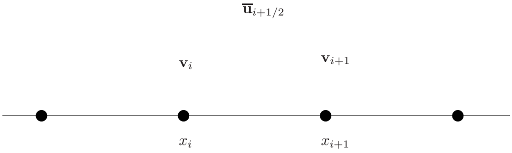

Idea

Given a problem (say compressible)

- Consider 2 versions

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{u})}{\partial x} = 0 \qquad \text{and } \frac{\partial \mathbf{v}}{\partial t} + A\frac{\partial \mathbf{v}}{\partial x} = 0$$

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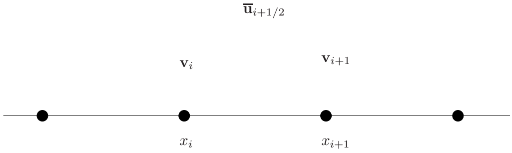

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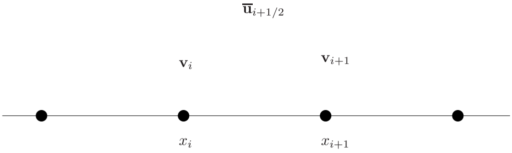

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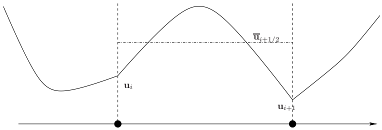
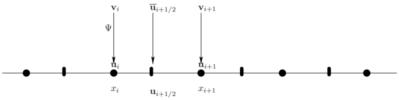
- Use the non conservative form at the grid points
- Reminiscient of Active Flux (Roe) if use conservative form at the vertices





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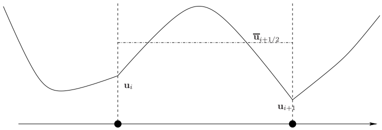

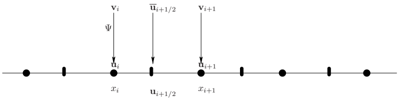

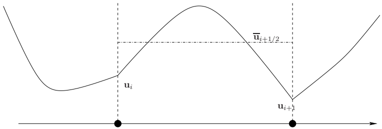

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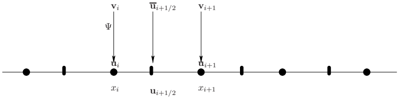

Quadratic R_u: $R_u(x_i) = \mathbf{u}_i, R_u(x_{i+1}) = \mathbf{u}_{i+1},$

$$\int_{x_i}^{x_{i+1}} R_u(x)dx = \Delta x_{i+1/2}\overline{\mathbf{u}}_{i+1/2}$$

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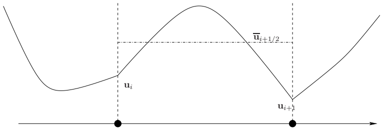

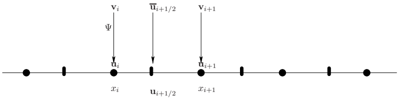

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Simpson formula: $\overline{\mathbf{u}}_{i+1/2} = \frac{1}{6}\big(\mathbf{u}_i + 4\mathbf{u}_{i+1/2} + \mathbf{u}_{i+1}\big)$

First order in time

- Average values:

$$\overline{\mathbf{u}}_{i+1/2}^{n+1} = \overline{\mathbf{u}}_{i+1/2}^{n} - \frac{\Delta t}{\Delta x_{i+1/2}} \big(\mathbf{f}(\mathbf{u}_{i+1}) - \mathbf{f}(\mathbf{u}_i)\big).$$

- System at point values

$$\mathbf{v}_i^{n+1} - \mathbf{v}_i^n + \Delta t \left(\overleftarrow{\Phi}_{i+1/2} + \overrightarrow{\Phi}_{i-1/2} \right) = 0$$

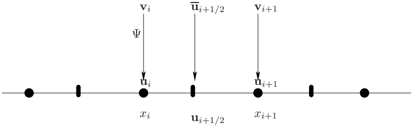
- $\overleftarrow{\Phi}_{i+1/2}$ is a (for example) consistent approximation of $A^- \dfrac{\partial \mathbf{v}}{\partial x}$ using data on $[x_i, x_{i+1/2}]$
- $\overrightarrow{\Phi}_{i+1/2}$ is a (for example) consistent approximation of $A^+ \dfrac{\partial \mathbf{v}}{\partial x}$ using data on $[x_i, x_{i+1/2}]$

Two examples

- First order

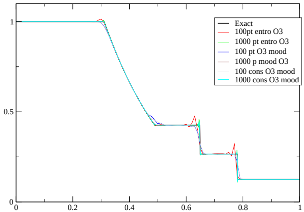
$$\overleftarrow{\Phi}_{i+1/2} = A^{-}(\mathbf{v}_i)\frac{\mathbf{v}_i - \mathbf{v}_{i+1/2}}{\Delta_{i+1/2}x/2}, \overrightarrow{\Phi}_{i+1/2} = A^{+}(\mathbf{v}_i)\frac{\mathbf{v}_{i-1/2} - \mathbf{v}_i}{\Delta_{i-1/2}x/2}$$

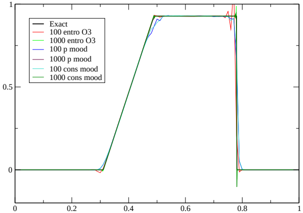
- Second and higher order: Based on Iserle's paper (IMA J. Num. Anal, 1981

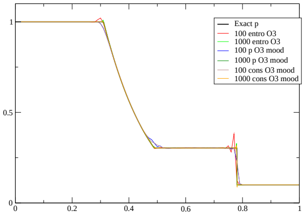


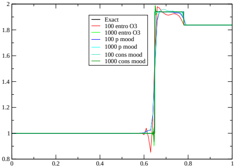
Some numerical illustrations

Sod

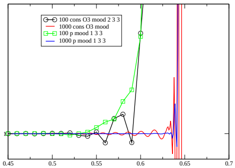






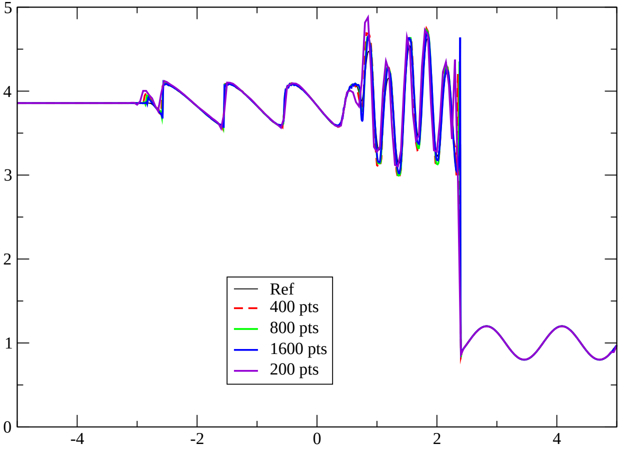


Exact
100 entro O3
1000 entro O3
100 p mood
1000 p mood
100 cons mood
1000 cons mood



100 cons O3 mood 2 3 3
1000 cons O3 mood
100 p mood 1 3 3
1000 p mood 1 3 3

Shu-Osher



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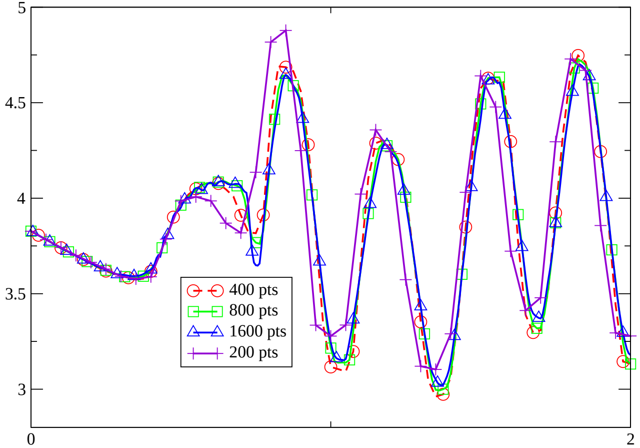

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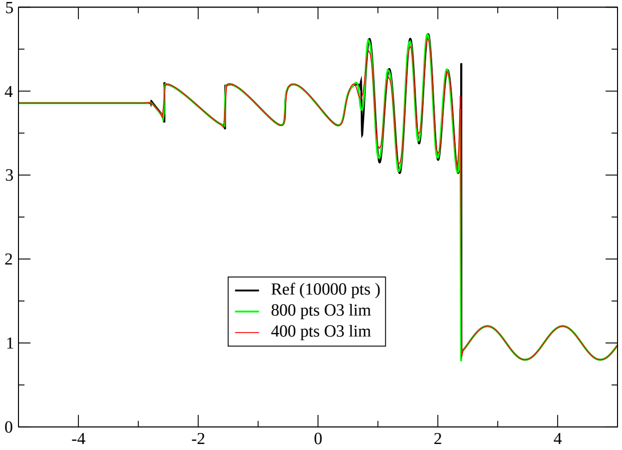
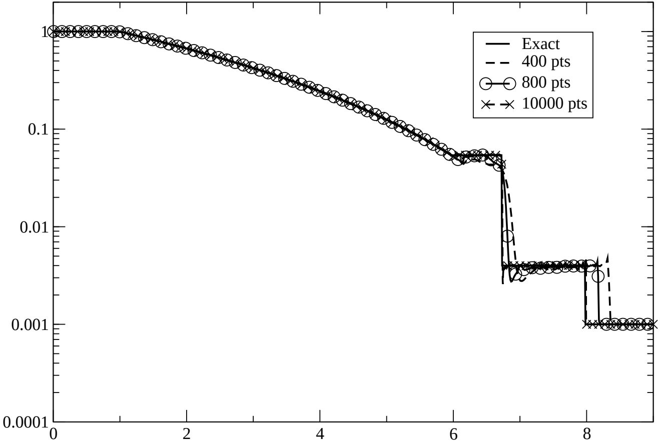


LeBlanc



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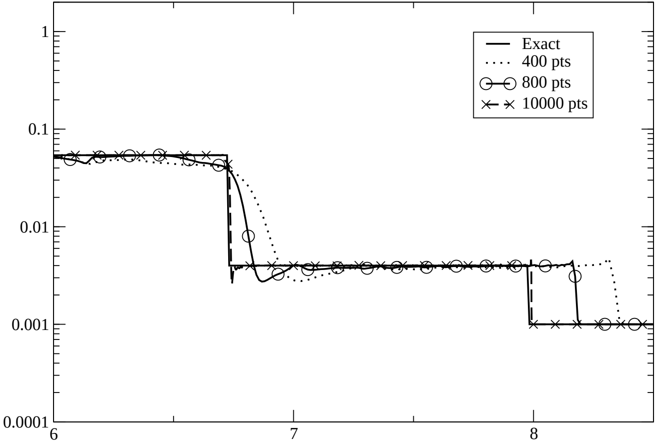

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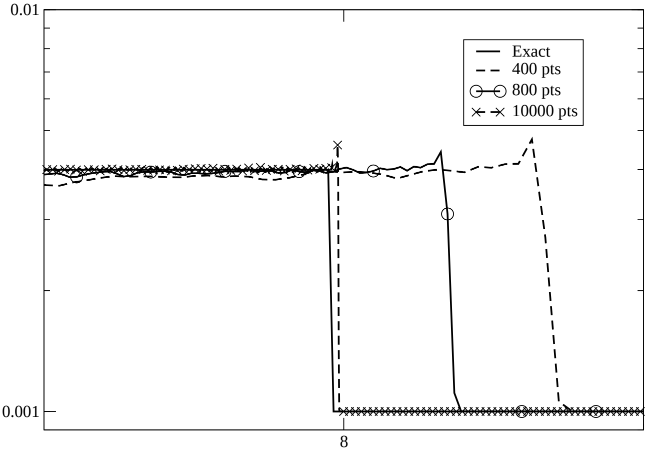

Why does it works?

$$\bar{\mathbf{u}}_{j+1/2}^{n+1} = \bar{\mathbf{u}}_{j+1/2}^{n} - \frac{\Delta t_n}{\Delta_{j+1/2}} \big(\underbrace{\mathbf{f}(\mathbf{u}_{j+1}^n) - \mathbf{f}(\mathbf{u}_j^n)}_{:=\delta_{j+1/2}\mathbf{f}} \big),$$

and

$$\begin{aligned} \mathbf{v}_j^{n+1} &= \mathbf{v}_j^n - \Delta t_n \big(\overleftarrow{\Phi}^{\mathbf{v}}_{j+1/2} + \overrightarrow{\Phi}^{\mathbf{v}}_{j-1/2} \big) \\ &= \mathbf{v}_j^n - \frac{\Delta t_n}{\Delta_j} \delta_x \mathbf{v}_j, \qquad \Delta_j = \frac{\Delta_{j+1/2} + \Delta_{j-1/2}}{2} \end{aligned}$$

Using $\mathbf{u}_j^{n+1} = \Psi(\mathbf{v}_j^{n+1})$, we get

$$\Delta_j \big(\mathbf{u}_j^{n+1} - \mathbf{u}_j^n \big) + \Delta t_n \delta_x \mathbf{u}_j = 0$$

with

$$\delta_x \mathbf{u}_j = \frac{\Delta_j}{\Delta t_n} \left(\Psi(\mathbf{v}_j^n - \frac{\Delta t_n}{\Delta_j} \delta \mathbf{v}_j) - \Psi(\mathbf{v}_j^n) \right),$$

which, thanks to the assumptions we have made on Ψ + Lipschitz continuity of the $\overleftarrow{\Phi}$ and $\overrightarrow{\Phi}$:

$$\|\delta_x \mathbf{u}_{j+1/2}\| \leq C \|\delta_x \mathbf{v}_j\| \leq C \sum_{j=-p}^{p} \|\mathbf{v}_{j+l} - \mathbf{v}_{j+l+1}\| \leq C \sum_{l=-p}^{p} \|\mathbf{u}_{j+l} - \mathbf{u}_{j+1+l}\|$$

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We have

$$\int_{\mathbb{R}} \varphi(x,t_n)(\mathbf{u}(x,t_{n+1}-\mathbf{u}(x,t_n))\ dx = \sum_{i\in\mathbb{Z}} \int_{x_i}^{x_{i+1}} \varphi(x,t_n)(\mathbf{u}(x,t_{n+1}-\mathbf{u}(x,t_n))\ dx$$

$$\approx \sum_{i\in\mathbb{Z}} \frac{\Delta_{j+1/2}}{6}\Big(\varphi(x_i,t_n)(\mathbf{u}_i^n-\mathbf{u}_i^{n+1}) + 4\varphi(x_{i+1/2}t_n)(\mathbf{u}_{i+1/2}^n-\mathbf{u}_{i+1/2}^{n+1})$$

$$+\varphi(x_{i+1},t_n)(\mathbf{u}_{i+1}^n-\mathbf{u}_{i+1}^{n+1})\Big)$$

setting $\delta_j^{n+1/2}\mathbf{u} = \mathbf{u}_j^{n+1} - \mathbf{u}_j^n$, $\Delta_{j+1/2} = x_{j+1} - x_j$, we get

$$\sum_{[x_j,x_{j+1}],j\in\mathbb{Z}} \frac{\Delta_{j+1/2}}{6}\left(\varphi_{j+1}^n\delta_j^{n+1/2}\mathbf{u} + 4\varphi_{j+1/2}^n\delta_{j+1/2}^{n+1/2}\mathbf{u} + \varphi_j^n\delta_j^{n+1/2}\mathbf{u}\right)$$

$$= \sum_{[x_j,x_{j+1}],j\in\mathbb{Z}} \Delta_{j+1/2}\varphi_{j+1/2}^n\delta_{j+1/2}^{n+1/2}\bar{\mathbf{u}}$$

$$+ \underbrace{\sum_{j\in\mathbb{Z}} \frac{\delta_j^{n+1/2}\mathbf{u}}{6}\left\{\Delta_{j+1/2}(\varphi_j - \varphi_{j+1/2}) + \Delta_{j-1/2}(\varphi_j - \varphi_{j-1/2})\right\}}_{S_n}.$$

so that we get

$$\sum_{n\in\mathbb{N}}\sum_{[x_j,x_{j+1}],j\in\mathbb{Z}} \frac{\Delta_{j+1/2}}{6}\left(\varphi_{j+1}^n\delta_j^{n+1/2}\mathbf{u} + 4\varphi_{j+1/2}^n\delta_{j+1/2}^{n+1/2}\mathbf{u} + \varphi_j^n\delta_j^{n+1/2}\mathbf{u}\right)$$

$$+ \Delta t_n \sum_{n\in\mathbb{N}}\sum_{[x_j,x_{j+1}],j\in\mathbb{Z}} \varphi_{j+1/2}^n\delta_{j+1/2}\mathbf{f}$$

$$+ \sum_{n\in\mathbb{N}} \Delta t_n S_n = 0 \qquad \delta_{j+1/2}\mathbf{f} = \mathbf{f}(\mathbf{u}_{j+1}) - \mathbf{f}(\mathbf{u}_j)$$

Conclude by

- Standard arguments for

$$\sum_{n\in\mathbb{N}} \sum_{[x_j,x_{j+1}],j\in\mathbb{Z}} \frac{\Delta_{j+1/2}}{6}\Big(\varphi_{j+1}^n \delta_j^{n+1/2}\mathbf{u} + 4\varphi_{j+1/2}^n \delta_{j+1/2}^{n+1/2}\mathbf{u} + \varphi_j^n \delta_j^{n+1/2}\mathbf{u}\Big)$$

  and

$$\sum_{n\in\mathbb{N}} \Delta t_n \sum_{[x_j,x_{j+1}],j\in\mathbb{Z}} \varphi_{j+1/2}^n \delta_{j+1/2}\mathbf{f}$$

- Weak BV estimates for

$$\sum_{n\in\mathbb{N}} \Delta t_n S_n \to 0$$

Ref: R. Abgrall, arxiv `https://arxiv.org/abs/2011.12572`, submitted 2020

Overview

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